1. (a) \( u(x,y) = \sqrt{1 + xy} \), \( v(x,y) = 0 \).

\[ u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = 0. \]

\[ u_y(0,0) = \lim_{h \to 0} \frac{u(0,h) - u(0,0)}{h} = 0. \]

So C/R hold at \((0,0)\).

(b) Consider \( \Delta z = h(1+i) \).

Then \[ \frac{f(0+\Delta z) - f(0)}{\Delta z} = \frac{\sqrt{|h(1+i)|}}{h(1+i)} \]

\[ = \frac{|h|}{h} \cdot \frac{1}{1+i} \] does not approach a limit as \( h \to 0 \) \((\Rightarrow \frac{1}{1+i} = 1+i \) as \( h \to 0^+ \), \( \Rightarrow \frac{-1}{1+i} = h \to 0^- \)).

Hence \( f'(0) \) can't exist.

(c) C/R is necessary but not sufficient for differentiability, so no contradiction.
Q2

\[ \text{on } C_1: z(t) = -2 \text{e}^t, \quad 0 \leq t \leq 1 \]
\[ \Rightarrow z'(t) = -2 \text{e}^t, \quad \int_{C_1} \text{Im}(z^2) \, dz = \int_0^1 -2 \text{e}^t \, dt = \frac{-2 \text{e}^t}{2} \bigg|_0^1 = -1 \]
\[ \text{on } C_2: z(t) = t - i, \quad -2 \leq t \leq 2 \]
\[ \Rightarrow z'(t) = 1, \quad \int_{C_2} \text{Im}(z^2) \, dz = \int_{-2}^2 \text{Im}(z^2) \, dt = 4 \]
\[ \text{on } C_3: z(t) = 2 + it, \quad -1 \leq t \leq 0 \]
\[ \Rightarrow z'(t) = i, \quad \int_{C_3} \text{Im}(z^2) \, dz = \int_{-1}^0 \text{Im}(z^2) \, dt = -1 \]

b) \[ \text{on } C: z = \text{e}^{i\theta}, \quad -\pi \leq \theta \leq 0 \]
\[ z' = i \text{e}^{i\theta} \]
\[ \Rightarrow \int_{C} \text{Im}(z^2) \, dz = 2 \int_{-\pi}^0 \text{Im}(\text{e}^{2i\theta}) \, d\theta = 2 \int_{-\pi}^0 \sin 2\theta \, d\theta = -4 \int_{-\pi}^0 \sin^2 \theta \, d\theta = -4 \int_{-\pi}^0 \frac{1 - \cos 2\theta}{2} \, d\theta = 2 \int_{-\pi}^{\pi} \sin 2\theta \, d\theta = 0 \]
\[ \neq 2 \int_{-\pi}^0 (-\cos \theta)^\frac{1}{2} - 2(\theta - \frac{\sin^2 \theta}{2}) \, d\theta = -2 \pi \]

C) \[ \text{on } C: z = 2 \text{e}^{i\theta}, \quad -\pi \leq \theta \leq 0 \]
\[ \Rightarrow z' = -2i \text{e}^{i\theta} \]
\[ \Rightarrow \int_{C} \text{Im}(z^2) \, dz = 2 \int_{-\pi}^0 \text{Im}(\text{e}^{2i\theta}) \, d\theta = 2i \int_{-\pi}^{\pi} \sin 2\theta \, d\theta = 0 \]
\[ = 4i \int_{-\pi}^0 \sin \theta \cos \theta \, d\theta + 4 \int_{-\pi}^0 \sin^2 \theta \, d\theta = -2 \pi \]
(1) \( f \) is not analytic on any domain containing any 2 of the curves from (a), (b) \& (c) by the Theorem from class (§43), if it were, the integrals would have to be equal.

(Note that \( f \) is in fact nowhere analytic.)

(2) a) The integrand is analytic on \( C \), with primitive \( e^z - \log z \); here \( \log z \) is a branch of the logarithm chosen with branch cut on the negative \( \text{Im} \) axis, i.e.,

\[
\log(re^{i\theta}) = \ln r + i\theta, \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}.
\]

So

\[
\int_C \left( e^z - \frac{1}{2} \right) \, dz = e^z - \log z \bigg|_a^b,
\]

\[
= \frac{1}{e} - \pi i - e + 2\pi i
\]

\[= \frac{1}{e} - e + \pi i.
\]

b) Integrand is analytic, so integral is path independent. A primitive is \( \sinh z \).

So

\[
\int_{0i}^{2\pi i} \cosh z \, dz = \sinh z \bigg|_{0i}^{2\pi i} = 0.
\]
(4) \( \frac{1}{z^2 + 4} \) is analytic on \( \mathbb{C} \setminus \{ \pm 2i \} \). In particular, it is analytic on \( \mathbb{C} \).

Since \( \frac{1}{z^2 + 4} = \frac{-\frac{1}{4}i}{z + 2i} - \frac{-\frac{1}{4}i}{z - 2i} \),

\( \Rightarrow \; \int_{C} \frac{dz}{z^2 + 4} = -\frac{1}{4}i \int_{C} \frac{dz}{z + 2i} + \frac{1}{4}i \int_{C} \frac{dz}{z - 2i} \).

as long as the integrals on the RHS exist.

Now, we distinguish 3 cases (note \( C \subset D \Rightarrow \) there are the only possible cases):

A. \( 2i, -2i \notin \text{Int} \mathbb{C} \), \( C \) trivially oriented

B. \( 2i \in \text{Int} C \), \( -2i \notin \text{Int} C \)

C. \( 2i, -2i \notin \text{Int} C \).

In (A), Cauchy integral formula \( \Rightarrow \) \( \text{II} = \text{III} = 2\pi i \), so \( I = 0 \).

Similarly in (B), \( \text{II} = \text{III} = -2\pi i \).

In (C), Cauchy-Goursat \( \Rightarrow \) \( \text{II} = \text{III} = 0 \) \( \Rightarrow I = 0 \).
5) Put \( f(z) = \frac{z^2 + 8z + 42}{(z^2 + 4)(z^2 - 4z + 5)} \)

\[
|f(z)| = \frac{|1 + \frac{8}{z} + \frac{42}{z^2}|}{|1 + \frac{4}{z^2}| |z^2 - 4z + 5|}
\]

\[
\Rightarrow \frac{1}{|z|^2} \quad \text{as} \quad |z| \to \infty.
\]

In particular, on \( C_R \), \( |f(z)| < \frac{2}{R^2} = M_R \)
for \( R \) suff. large.

Since \( L_R = \text{length} \ C_R = 2\pi R \), we have via M-\text{\&} \text{L} \[
\int_{C_R} f(z) \, dz \leq L_R M_R = \frac{4\pi}{R} \to 0
\]
as \( R \to \infty \).

6) \( f(z) = \sin z \) is analytic on \( C \). So by Cauchy's formula:

\[
f^{(6)}(-1) = \frac{6!}{2\pi i} \int_C \frac{\sin z \, dz}{(z+1)^6}.
\]

\[
\int_C \frac{\sin z \, dz}{(z+1)^6} = 2\pi i \left. \frac{d^6}{dx^6} \sin(x) \right|_{x=-1}
\]

\[
= \frac{2\pi i}{6!} (-\sin(-1))
\]

\[
= \frac{2\pi i (\sin 1)}{6!} \quad (\approx 0.0093).
\]