MATH3401/3901 2023 sol^{n}s.

1) a) $w = \tanh^{-1} z \Rightarrow z = \tanh w = \dfrac{\sinh w}{\cosh w}$

$$= \frac{(e^{w} - e^{-w})/2}{(e^{w} + e^{-w})/2} \quad 1 \quad = \frac{e^{2w} - 1}{e^{2w} + 1}$$

$$\Rightarrow (e^{2w} + 1) z = e^{2w} - 1$$

$$\Rightarrow e^{2w}(z - 1) = -1 - z$$

$$\Rightarrow e^{2w} = \frac{1+z}{1-z} \quad 2 \qquad z \neq 1$$

$$\Rightarrow w = \frac{1}{2} \log \frac{1+z}{1-z} \quad 1 \qquad z \neq 1, -1 \quad 1$$

b) $\tanh^{-1} i = \frac{1}{2} \log \frac{1+i}{1-i}$ 1

2
$$= \frac{1}{2} \log \frac{(1+i)^2}{(1-i)(1+i)}$$

$$= \frac{1}{2} \log \frac{2i}{2}$$

$$= \frac{1}{2} \log i = \frac{1}{2}\left(\ln |i| + i \arg i\right) \quad 1$$

$$= \frac{i}{2}\left(\frac{\pi}{2} + 2n\pi\right) \qquad n \in \mathbb{Z}$$

$$= i\left(\frac{\pi}{4} + n\pi\right) \qquad n \in \mathbb{Z}$$

1

(2) a) C/R is necessary for diff-bility (1)

C/R v.2 (using Wirtinger) $\Rightarrow$

$$\frac{\partial f}{\partial \bar{z}} = 0 \iff 3\bar{z}^2 + 3 = 0 \iff \bar{z} = \pm i \iff z = \pm i.$$ * (2)

Since u & v are poly f^n's of x & y, f satisfies the sufficient cond^n-s for diff-bility at $\pm i$. (1)

Hence, f is diff-ble at $\pm i$ (only). (1)
diffble

b) f is not ~~analytic~~ on a nbhd of any pt in $\mathbb{C}$, so is not analytic anywhere. (1.5) (1)

c) Analytic $\Rightarrow$ differentiable, but differentiable $\not\Rightarrow$ analytic. (2.5)

Note re: a) alternatively: (*) OR

$$f(z) = (x-iy)^3 + 3(x-iy)$$
$$= x^3 - 3ix^2y - 3xy^2 + iy^3 + 3x - 3iy$$
$$= u+iv, \quad u = x^3 - 3xy^2 + 3x \ \& \ v = -3x^2y + y^3 - 3y$$

$\Rightarrow u_x = 3x^2 - 3y^2 + 3 \qquad u_y = -6xy$

$v_x = -6xy \qquad v_y = -3x^2 + 3y^2 - 3$

C/R I $\quad u_x = v_y \Rightarrow 6x^2 - 6y^2 + 6 = 0 \quad$ (I)

C/R II $\quad u_y = -v_x \Rightarrow -6xy = 6xy \quad$ (1)

$\Rightarrow -12xy = 0 \Rightarrow x = 0$ or $y = 0$. (II)

$x = 0$ in (I) $\Rightarrow y = \pm 1 \Rightarrow \underline{z = \pm i}$ (1)

$y = 0$ in (I) $\Rightarrow$ no sol^n.

③ $\frac{1}{z^2+1}$ is analytic on $\mathbb{C} \setminus \{\pm i\}$. In particular, it is analytic on C. } ①

Since $C \subset D$, one of 3 cases occurs:

Ⓘ $i, -i \in \operatorname{Int} C$, C +vely oriented;

Ⓘⓘ " " " -vely " ;

Ⓘⓘⓘ $i, -i \notin \operatorname{Int} C$.

In case Ⓘⓘⓘ, Cauchy Goursat $\Rightarrow$ result. ②

Otherwise, note via partial fractions:

$$\frac{1}{z^2+1} = \frac{a}{z+i} + \frac{b}{z-i}$$

$$= \frac{az - ai + bz + bi}{z^2+1}$$ ②

$\Rightarrow az - ai + bz + bi = 1$.

$z = i \Rightarrow 2bi = 1 \Rightarrow b = -i/2$

$z = -i \Rightarrow -2ai = 1 \Rightarrow a = i/2$ ①

So $\displaystyle\int_C \frac{dz}{z^2+1} = \frac{i}{2}\underbrace{\int_C \frac{dz}{z+i}}_{\mathrm{I}} + \frac{-i}{2}\underbrace{\int_C \frac{dz}{z-i}}_{\mathrm{II}}$, as

long as the integrals on the RHS exist. ②

Case I: Cauchy integral formula $\Rightarrow$ I = II $= 2\pi i \Rightarrow$ result.

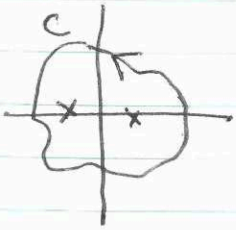
Case II: " " " " $= -2\pi i$ " " ②

OR use residue thm for cases I, II.

need to reallocate in that case

using res th^{m}.

Case II



$$f(z) = \frac{1}{z^2+1}$$

$$\int_C f(z)\,dz = 2\pi i \Big(\underbrace{\operatorname*{res}_{z=i} f}_{A} + \underbrace{\operatorname*{res}_{z=-i} f}_{B} \Big) \quad (*)$$

1

for A: near i $\quad f(z) = \frac{1}{z^2+1} = \frac{p}{q}$

$p(i) = 1$, $q(i) = 0$, $q'(i) = 2i \neq 0$ 1

$\Rightarrow$ simple pole, $\quad A = \frac{p(i)}{q'(i)} = \frac{1}{2i} = -i/2$

for B: near $-i$ same p, q:

$$B = \frac{p(-i)}{q'(-i)} = \frac{1}{-2i} = i/2 \quad 1$$

$(*) \Rightarrow \int_C f(z) = 0$.

Case I $\quad$ follows from case II, noting $\int_C f = -\int_{-C} f$. 1

IF students do both cases in full, suggest re-allocating 2 points per case.

4 a) $u = \sin(x-1)\cosh y$

(1) $u_x = \cos(x-1)\cosh y \qquad u_{xx} = -\sin(x-1)\cosh y$ (1)

$u_y = \sin(x-1)\sinh y \qquad u_{yy} = \sin(x-1)\cosh y.$

$\Rightarrow u_{xx} + u_{yy} = 0.$

Further, u & partials of all orders are cts (products of $\cos(x-1)/\sin(x-1)/\cosh y/\sinh y$) (1)

Hence, u is harmonic on $\mathbb{R}^2$.

b) $C/R_I \Rightarrow u_x = v_y \Rightarrow v_y = \cos(x-1)\cosh y$

$\Rightarrow v = \int \cos(x-1)\cosh y \, dy$ (1)

$= \cos(x-1)\sinh y + \phi(x)$ (*)

$C/R_{II} \Rightarrow v_x = -u_y.$

Comparing $(*)_x$ & u_y from a), this yields:

$-\sin(x-1)\sinh y + \phi'(x) = -\sin(x-1)\sinh y$ (1)

$\Rightarrow \phi'(x) = 0$, so e.g. $\phi = 0$. (1)

So from (*), $v = \cos(x-1)\sinh y$ is a harmonic conjugate of u.

c) $f(z) = u + iv = \sin(x-1)\cosh y + i\cos(x-1)\sinh y$

$= \sin(x-1)\cos(iy) + i\cos(x-1)(-i)\sin(iy)$

$= \sin(x-1)\cos(iy) + \cos(x-1)\sin(iy)$

$= \sin(x-1+iy)$

$= \sin(z-1).$ (3) f? must be of z

d) g is entire $\Rightarrow w$ is (also) a harmonic conjugate of u, so $v - w = \text{constant } (\in \mathbb{R})$. (1) (2)

5 a) (i) f is a quotient of entire f^n s, so singularities are zeros of denominator $= 0$.

$$f(z) = \frac{1}{z^4}\left(1 + z^2 + \frac{z^4}{2!} + \cdots\right)\left(6z - \frac{(6z)^3}{3!} + \frac{(6z)^5}{5!}\right) \quad 1$$

$$= \frac{1}{z^4}\left(6z + (6-36)z^3 + \left(3 + \frac{6^5}{5!} - 36\right)z^5 + \cdots\right)$$

$$= \frac{6}{z^3} - \frac{30}{z} + 31.8z + \cdots \quad 1$$

(ii) from Laurent series, f ~~is~~ has a pole of order 3 at 1 0 (largest −ve power) 1

(iii) res = coeff of 1 $z^{-1} = -30$ 1

b) Exponent $(1/z^2)$ undefined at 0, so singularity.

(i) $$f(z) = \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!} = 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \frac{1}{3!z^6} + \cdots \quad 2$$

(ii) from Laurent series, f has an essential singularity at 1 0 (arbitrarily large 1 −ve powers of z)

(iii) res = coeff of $z^{-1} = 0$. 1 1

(6) a) $$PV\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx. \quad (*)$$ 1

b) Put $f(x) = \dfrac{x^7}{(x^4+1)^2}$. f is defined on all of $\mathbb{R}$, & is odd. Hence 1

$$\int_{-R}^{0} f = -\int_{0}^{R} f. \quad \text{Hence } \int_{-R}^{R} f = 0 \ \forall R$$ 1

So from $(*)$, $PV\int_{-\infty}^{\infty} f = 0$.

c) $f \sim 1/x$ as $x \to \pm\infty$, so $\lim_{M\to\infty}\int_0^M f$, 1

$\lim_{N\to-\infty}\int_N^0 f$ are undefined, hence 1

$\int_{-\infty}^{\infty} f$ is undefined. 1

(7)

a) $f(z)$ is analytic on $\mathbb{C} \setminus \{$zeros of denominator of $f\}$ (1)

zeros of z^4+1 are 4th roots of $e^{i\pi}$

$= e^{i\pi/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}$. (1)

$= \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}$

$= z_1, z_2, z_3, z_4$. (1)

or other argument

Each of these is a simple pole, since $f(z)$ can be written as $\frac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$, and z^2 doesn't vanish at z_1, z_2, z_3, z_4. (1)

b) At both z_1 & z_2, we can write $f(z)$ as

(2) $\frac{p(z)}{q(z)}$, where $p(z) = z^2 \neq 0$ & $q(z) = z^4+1$ a simple zero at z_j. Hence the residue is

$$\frac{p(z)}{q'(z)} = \frac{z_j^2}{4z_j^3} = \frac{1}{4z_j} = \frac{1}{4}\bar{z}_j$$ (because z_j are on the unit circle). (2)

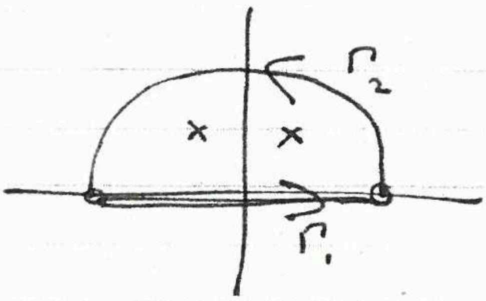
So: $\operatorname{res}_{z=\frac{1+i}{\sqrt{2}}} f(z) = \frac{1-i}{4\sqrt{2}}$ & $\operatorname{res}_{z=\frac{-1+i}{\sqrt{2}}} \frac{-1-i}{4\sqrt{2}}$ as req^d.

(1)

Other methods poss.

(7) c) Consider contour as below:

$\Gamma = \Gamma_1 + \Gamma_2$



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Note that f is analytic inside & on Γ, except for isolated singularities at z_1, z_2.
Hence,

(7) ctd.

Residue Th^{m} $\Rightarrow$ $\int_{\Gamma} f = 2\pi i \sum_{j=1}^{2} \operatorname{res}(f, r_j)$. (2)

$RHS \overset{(b)}{=} \left(\frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}}\right) 2\pi i = \frac{\pi}{\sqrt{2}}$ (B) (1)

$LHS = \underbrace{\int_{-R}^{R} f(x)dx}_{I} + \underbrace{\int_{\Gamma_2} f(z)dz}_{II}$. (A) As $R \to \infty$,

$I \Rightarrow PV\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx$, since f is even. (1) (1)

$|II| \leq$ ~~$\int \frac{|z|^2}{|z^4+1|}dz$~~ Ml, (*) where M is a bound for the modulus of the integrand, & $l = l(\Gamma_2) = \pi R$.

To estimate M, observe (1)

$$|f| = \frac{|z|^2}{|z^4+1|} \leq \frac{R^2}{R^4-1} \quad \text{since } |z| = R \text{ on } \Gamma_2.$$

Hence (*) $\Rightarrow \int_{\Gamma_2} |f| \leq \frac{\pi R^3}{R^4-1} = \frac{\pi}{R - 1/R^3} \to 0$ as $R \to \infty$. (1) (1)

Combining this in (A) with (B) gives the desired result. (1)

8 a) For $z = x+iy$, there holds:

$$
\left.\begin{aligned}
|e^z| &= |e^{x+iy}| \\
&= |e^x e^{iy}| \\
&= |e^x|\cdot|e^{iy}| \\
&= e^x \cdot 1 \\
&= e^x
\end{aligned}\right\} \text{(1)}
$$

$$
\left.\begin{aligned}
&\leq e^{\sqrt{x^2}} \\
&\leq e^{\sqrt{x^2+y^2}}
\end{aligned}\right\} \text{(1)}
$$

$$= e^{|z|}.$$

Can be shorter for full marks!

0.5 1

b) Let $f(z) = 4z^n$ & $g(z) = e^z + 1$. f & g are entire.

On $\{z : |z| = 1\}$: $|g(z)| = |e^z + 1| \leq |e^z| + 1$ (1)

$\leq e^{|z|} + 1$ by a) (1) $= e + 1 \leq 4 = 4 \cdot 1^n = |f(z)|$

Hence by Rouché, f & $f+g = h$ have the same number of zeros (counting multiplicity) inside $\{z : |z| < 1\}$, namely n, since f has an n-fold zero at 0. (1.5)

(1)

c) No. Consider e.g. $z = iy$: (2)

$h(z) = e^{iy} + 4(iy)^n + 1$, and:

$|e^{iy}| = 1 \ \forall y$, $|1| = 1$, & $|4(iy)^n| = 4|y|^n \to \infty$ as $|y| \to \infty$. Hence $|h(z)|$ is unbounded on the imaginary axis, and hence on $\mathbb{C}$.

answer "no example" without explanation $\Rightarrow$ $\frac{1}{2}/2$

(9) a) Let $f(z) = [\mathrm{Re}(z)]^2$ So $u = x^2$, $v = 0$.

$u_x = 2x$, $u_y = v_x = v_y = 0$.

$C/R_I \Rightarrow u_x = v_y \Rightarrow 2x = 0$ (*)

$C/R_{II} \Rightarrow v_y = -u_x \Rightarrow 0 = 0$.

Solving (*), C/R are satisfied precisely on the imaginary axis.

Since u, v & partials are cts, f is differentiable on the Imaginary axis.

b) none exists : f is ~~only~~ not diff^{ble} on a nbhd of any point.

c) None exist. If such an f existed $g(z) = 1/f(z)$ would be non-constant, bdd & entire : no such f^{n} by Liouville.

d) No such f^{n} : $T(1) = T(-i)$ & Möb. transf are 1-1.

e) $T(z) = z$.