MATH3401 2024

Q1 a) Want to solve $z = \tan w = \dfrac{\sin w}{\cos w}$ for w.

So $z = \dfrac{\sin w}{\cos w} = \dfrac{\dfrac{e^{iw} - e^{-iw}}{2i}}{\dfrac{e^{iw} + e^{-iw}}{2}} = \dfrac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})}$

$$= \frac{e^{2iw} - 1}{i(e^{2iw} + 1)}$$

$\iff zi(e^{2iw} + 1) = e^{2iw} - 1$

$\iff e^{2iw}(zi - 1) = -(zi + 1)$

$\iff e^{2iw} = \dfrac{1 + zi}{1 - zi} = \dfrac{i - z}{i + z}$ for $z = -i$

$\iff 2iw = \log\left(\dfrac{i-z}{i+z}\right)$ for $z \neq \pm i$

$\iff w = \dfrac{1}{2i}\log\left(\dfrac{i-z}{i+z}\right) = \dfrac{-1}{2i}\log\left(\dfrac{i+z}{i-z}\right) = \dfrac{i}{2}\log\left(\dfrac{i+z}{i-z}\right)$,

for $z \neq \pm i$, as req^d.

b) $\tan z = i/2 \Rightarrow z = i/2 \log\left(\dfrac{i + i/2}{i - i/2}\right)$

$= i/2 \log 3$

$= i/2 (\ln|3| + i\arg(3))$

$= i/2 (\ln 3 + i \cdot 2n\pi)$ $n \in \mathbb{Z}$

$= \dfrac{i \ln 3}{2} + n\pi$ $n \in \mathbb{Z}$

Q2 $f(z) = |z|^6 = z^3 \bar{z}^3$

Note C/R are necessary for diff$^{\text{bility}}$.

C/R (Wirtinger) : $\frac{\partial f}{\partial \bar{z}} = 0 \iff 3 z^3 \bar{z}^2 = 0$

$\iff z = 0.$

Note $f(z) = |z|^6 = (x^2+y^2)^3 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6$

$= u + iv$ with $u = x^6 + 3x^4y^2 + 3x^2y^4 + y^6$

& $v = 0$.

Since u & v are polynomial f^n's of x & y, f satisfies the sufficient cond^n's for diff$^{\text{bility}}$ at 0.

Hence f is differentiable precisely at 0.

*Alternative to Wirtinger:

C/R_I $u_x = v_y \Rightarrow 6x^5 + 12x^3y^2 + 6xy^4 = 0$

$\Rightarrow 6x(x^4 + 2x^2y^2 + y^4) = 0$

$\Rightarrow 6x(x^2+y^2)^2 = 0$

$\Rightarrow x = 0.$

C/R_{II} $u_y = -v_x \Rightarrow 6x^4y + 12x^2y^3 + 6y^5 = 0$

$\Rightarrow 6y(x^2+y^2)^2 = 0$

$\Rightarrow y = 0$

Hence C/R_I, C/R_{II} only simultaneously hold at $z=0$.

b) f is not diff$^{\text{ble}}$ on a nbhd of any point in $\mathbb{C}$, so is not analytic anywhere.

c) Analytic $\Rightarrow$ differentiable, but differentiable $\not\Rightarrow$ analytic.

③ a) Put $f(z) = \dfrac{z^2+4z+7}{(z^2+4)(z^2+2z+2)}$

Note $|f(z)| = \left| \dfrac{1+ 4/z + 7/z^2}{(1+4/z^2)(z^2+2z+2)} \right|$

$\sim 1/|z|^2$ as $|z| \to \infty$.

In particular, on C_R, $|f(z)| < \frac{2}{R^2} = M_R$ for R suff. large.

Note L_R = length$(C_R) = 2\pi R$.

Hence (M-l estimate), $\left|\int_{C_R} f(z)\,dz\right| \leq L_R M_R = \frac{4\pi}{R} \to 0$ as $R \to \infty$.

b) f is a quotient of analytic f^ns. Hence, it is analytic except at the zeros of the denominator, which are $\pm 2i$, $-1 \pm i$, and hence lie inside C.
Thus f is analytic on C, C_R $(R>4)$ and the region inbetween. Cauchy-Goursat extension $\Rightarrow$

$$\int_C f = \int_{C_R} f.$$

Since the RHS $\to 0$ as $R \to 0$, the LHS (and indeed, the RHS) must be 0.

4 a) $u = e^{x-1}\cos y$

$u_x = e^{x-1}\cos y$ $\quad\quad$ $u_{xx} = e^{x-1}\cos y$

$u_y = -e^{x-1}\sin y$ $\quad\quad$ $u_{yy} = -e^{x-1}\cos y$

$\Rightarrow \Delta u = u_{xx} + u_{yy} = 0$

Further, partials of u of all orders are cts (products of e^{x-1} & $\cos/\sin y$)

Hence, u is harmonic on $\mathbb{R}^2$.

b) $C/R_{I} \Rightarrow u_x = v_y$

$\Rightarrow v_y = e^{x-1}\cos y$

$\Rightarrow v = \int e^{x-1}\cos y \, dy = e^{x-1}\sin y + \phi(x)$ ⊛

$C/R_{II} \Rightarrow v_x = -u_y$

Comparing ⊛$_x$ to $-u_y$ from a), this yields:

$e^{x-1}\sin y + \phi'(x) = e^{x-1}\sin y$

$\Rightarrow \phi' = 0$, so choose e.g. $\phi = 0$.

$\Rightarrow v = e^{x-1}\sin y$ is a harmonic conjugate of u.

c) $f(z) = e^{x-1}\cos y + i e^{x-1}\sin y$

$= e^{x-1+iy}$

$= e^{z-1}$.

d) g is entire $\Rightarrow w$ is also a harmonic conjugate of u, so $v - w =$ constant $(\in \mathbb{R})$.

(5) a)(i) f is a quotient of entire functions, so is singular at the zeros of the denominator, i.e. 0, which is hence an isolated singularity.

(ii) $$f(z) = \frac{1}{z^7}\left(1-\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)\right)^2$$
$$= \frac{1}{z^7}\left(\frac{z^2}{2}+\frac{z^4}{24}+\cdots\right)^2$$
$$= \frac{1}{z^7}\left(\frac{z^4}{4}+\frac{z^6}{24}+\cdots\right)$$
$$= \frac{1}{4z^3}+\frac{1}{24z}+\cdots$$

(iii) pole of order 3 (largest -ve power)

(iv) $\frac{1}{24}$ (coefficient of z^{-1}).

b)(i) $\frac{1}{z}$ is analytic on $\mathbb{C}_*$, so $\sin(1/z)$ is analytic on $\mathbb{C}_*$ (composition of analytic), so (since $42z^6$ is entire) $42z^6\sin(1/z)$ is analytic on $\mathbb{C}_*$ as a product of analytic f^ns. g is undefined at 0, since $\frac{1}{z}$ is, so g has an isolated sing at 0.

(ii) $$g(z) = 42z^6\left(\frac{1}{z}-\frac{1}{3!z^3}+\frac{1}{5!z^5}-\frac{1}{7!z^7}+\cdots\right)$$
$$= 42z^5 - 7z^3 + \frac{7}{20}z - \frac{1}{120z} + \cdots$$

(iii) essential singularity (arbitrarily large -ve powers of z)

(iv) $-\frac{1}{120}$ (coeff of z^{-1})

6 a)

<u>Note:</u> $f(z) = C_1 \sin(e^{-i\pi/4} z \pi)$ has zeros for $e^{-i\pi/4} z = n \in \mathbb{Z}$

i.e. $z = n e^{i\pi/4}$, so adjacent zeros are separated by unit distance. This f satisfies (i) & (ii)

for $z = \frac{e^{i\pi/4}}{2}$, we have $f(z) = C_1$.

So for $C_1 = 2$:

$f(z) = 2 \sin(e^{-i\pi/4} z \pi)$ suffices.

b) No. multiply e.g. by any f^n only vanishing at zeros of f, & $= 1$ at $\frac{e^{i\pi/4}}{2}$,

e.g. $g(z) = 2z / e^{i\pi/4}$:

$f \cdot g$ also satisfies the cond^{n}s (i)–(iii) f is a quotient of analytic f^{n}s, so

(a) Singularities are zeros of denominator $= \pm i, \pm 3i$.

Since we can write

$$f(z) = \frac{z^3 e^{iz}}{(z-i)(z+i)(z-3i)(z+3i)},$$

it follows that they are all simple poles, e.g.

$$f(z) = \frac{\phi(z)}{z-i},$$

$$\phi(z) = z^3 e^{iz} / (z+i)(z-3i)(z+3i) \quad \text{is analytic}$$

& nonzero near $z=i$ etc.

b) Since we have a simple pole at $z=i$,

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{i^3 e^{-1}}{(i+i)(-2i)(4i)}.$$

$$= \frac{-1}{16e}$$

Similarly for $z=3i$,

$$\operatorname*{Res}_{z=3i} f(z) = \left.\frac{z^3 e^{iz}}{(z-i)(z+i)(z+3i)}\right|_{z=3i} = \frac{27 i^3 e^{-3}}{(2i)(4i)(6i)}$$

$$= \frac{9}{16e^3}$$

$R >> 1$

c) 

Consider $\int_\Gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f$. (1)

Since f is analytic on $\mathbb{C} \setminus \{\pm i, \pm 3i\}$, we have via Cauchy residue thm & b)

$$\text{LHS} = 2\pi i \left(\frac{-1}{16e} + \frac{9}{16e^3} \right) \quad (2)$$

For $\int_{\gamma_2} f$, we note $f(z) = g(z)e^{iz}$,

where $g(z) = \frac{z^3}{(z^2+1)(z^2+9)}$ satisfies, for $z \in \gamma_2$ ($\Leftrightarrow |z|=R$):

$$|g(z)| \le \frac{R^3}{(R^2-1)(R^2-9)} \le \frac{R}{R^2/2 \; R^2/2}$$

for R sufficiently large $= \frac{4}{R}$, & g is analytic for $R>3$.

Hence by Jordan's lemma ($\alpha=1$, $M=4$, $\beta=1$) we see $\lim_{R\to\infty}\int_{\gamma_2} f = 0$. (3)

Taking the imaginary part of (1), we see via (2):

$$2\pi\left(\frac{-1}{16e}+\frac{9}{16e^3}\right) = \underbrace{\int_{-R}^{R}\frac{x^3\sin x}{(x^2+1)(x^2+3)}\,dx}_{\mathrm{I}} + \underbrace{\mathrm{Im}\int_{\gamma_2} f\,dx}_{\mathrm{II}}$$

i.e. $\frac{2\pi}{16e}\left(\frac{9}{e^2}-1\right) =$ " " (4)

since the integrand of (I) is even, by taking $\lim_{R\to\infty}$ in (4),

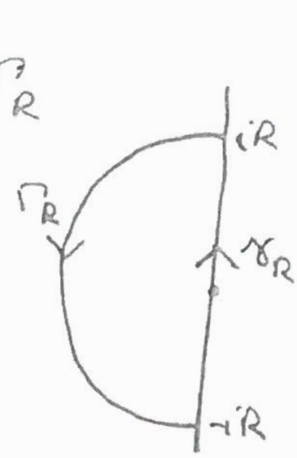
we see $$\frac{2\pi}{16e}\left(\frac{9}{e^2}-1\right) = \int_{-\infty}^{\infty}\frac{x^3\sin x}{(x^2+1)(x^2+3)}\,dx,$$

having used (3).

Since $\int_0^\infty \frac{x^3\sin x\,dx}{(x^2+1)(x^2+3)} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^3\sin x\,dx}{(x^2+1)(x^2+3)}$, this gives the desired result.

□

(8) Consider the contour $C_R = \gamma_R + \Gamma_R$ for R to be fixed later.



Put $f(z) = 2z+4$, $g(z) = 3e^z$

Note f, g entire.

On γ_R, $z = iy$, so $|f| = |4+2iy| \geq 4$, &

$|g| = 3|e^{iy}| = 3 < 4 \leq |f|$

On Γ_R, $z = x+iy$ with $x < 0$, $|z| = R$

$\Rightarrow |g| = 3|e^{x+iy}| = 3e^x < 3$ since $x < 0$

& $|f| > 2|z| - 4 = 2R - 4 > 3$ for $R > 7/2$

So $|f| > |g|$ for $R > 7/2$.

Hence, for $R > 7/2$, $|f| > |g|$ on C_R.

Rouché $\Rightarrow$ f & $f+g=h$ have the same # of zeros inside C_R. f has precisely 1 zero (at $z=-2$), so h does as well. Letting $R \to \infty$ $\Rightarrow$ h has one zero in the LHP $\Rightarrow$ result.

Q9 a) e.g. $f(z) = [\mathrm{Im}(z)]^2$.

For $z = x+iy$, $f(z) = y^2$

So $u = y^2$, $v = 0$

$u_x = v_x = v_y = 0$, $u_y = 2y$

$C/R_I \Rightarrow u_x = v_y = 0 \Rightarrow 0 = 0$

$C/R_{II} \Rightarrow u_y = -v_x = 2y = 0 \Rightarrow y = 0$ (*)

Solving (*), C/R hold precisely on the real axis. Since u, v & partials are cts, f is differentiable precisely on the real axis.

b) None exist. f is not differentiable on a nbhd of any point.

c) None exist. If such an f existed, $g(z) = 1/f(z)$ would be non-constant, bdd & entire. No such f^ns exist by Liouville

d) No such f^n : $T(0) = T(-i)$, and Mob transfs are 1-1

e) e.g. $T(z) = -iz$ $(a = -i, b = 0, c = 0, d = 1)$ does this.