MATH3401 2025, Sem 1, SOLUTIONS

Q1

a) $w = \coth^{-1} z \Rightarrow$

$$z = \coth w = \frac{\cosh w}{\sinh w} = \frac{(e^{w} + e^{-w})/2}{(e^{w} - e^{-w})/2} = \frac{e^{2w}+1}{e^{2w}-1}$$

$$\Rightarrow (e^{2w}-1)z = e^{2w}+1$$

$$\Rightarrow e^{2w}(z-1) = z+1$$

$$\Rightarrow e^{2w} = \frac{z+1}{z-1} \qquad z \neq 1$$

$$\Rightarrow w = \frac{1}{2}\log\frac{z+1}{z-1} \qquad z \neq 1, -1 \quad \text{as req}^{d}.$$

b)
$$\coth^{-1} i = \frac{1}{2}\log\frac{i+1}{i-1}$$

$$= \frac{1}{2}\log\frac{(i+1)^2}{(i-1)(i+1)}$$

$$= \frac{1}{2}\log\frac{2i}{-2}$$

$$= \frac{1}{2}\log(-i)$$

$$= \frac{1}{2}\Big[\underbrace{\ln|-i|}_{=0} + i\arg(-i)\Big]$$

$$= \frac{i}{2}\left(\frac{3\pi}{2} + 2n\pi\right) \qquad n \in \mathbb{Z}$$

$$= i\left(\frac{3\pi}{4} + n\pi\right) \qquad n \in \mathbb{Z}.$$

(2) a) set $h(z) = e^z \sinh z$
consider $\{z_n\}$, for $z_n = n\pi i$.
Then $\sinh z_n = 0 \Rightarrow h(z_n) = 0$.
Consider $\{w_n\}$ for $w_n = n$.
Then $h(w_n) = e^n \sinh n > e^n \cdot \frac{e^n}{2} = e^{2n}/2$.
Hence we have two sequences tending to ∞, for one of which $h \equiv 0$, and for one of which h is unbounded. Hence, the limit does not exist.

b) Claim $\lim_{z \to \infty} 1/e^{1/z} = 1$

$\Leftrightarrow \lim_{z \to 0} 1/e^{1/1/z} = 1$

$\Leftrightarrow \lim_{z \to 0} 1/e^z = 1$

$\Leftrightarrow 1/\lim_{z \to 0} e^z = 1$

$\Leftrightarrow 1 = 1$, which is true.

(3) $(1+z)^n = z^n$

$$\Rightarrow \left(\frac{1+z}{z}\right)^n = 1$$

$$\Rightarrow \left(1+\frac{1}{z}\right) = 1^{1/n} = \exp\frac{2\pi i k}{n} \qquad k=0,\dots,n-1 \quad (*)$$

No sol^{n} for $k=0$, since this leads to $\frac{1}{z}=0$, which has no solution.

Hence, there are $n-1$ solutions, given by $(*)$

as $$z_k = \frac{1}{\exp\frac{2\pi i k}{n} - 1} \qquad k=1,\dots,n-1.$$

So $$z_k = \frac{1}{\left(\cos\frac{2\pi k}{n}-1\right) + i\sin\frac{2\pi k}{n}} \cdot \frac{\left(\cos\frac{2\pi k}{n}-1\right) - i\sin\frac{2\pi k}{n}}{\left(\cos\frac{2\pi k}{n}-1\right) - i\sin\frac{2\pi k}{n}}$$

$$= \frac{\left(\cos\frac{2\pi k}{n}-1\right) - i\sin\frac{2\pi k}{n}}{\left(\cos\frac{2\pi k}{n}-1\right)^2 + \sin^2\frac{2\pi k}{n}}$$

$$= \frac{\left(\cos\frac{2\pi k}{n}-1\right) - i\sin\frac{2\pi k}{n}}{\cos^2\frac{2\pi k}{n} - 2\cos\frac{2\pi k}{n} + 1 + \sin^2\frac{2\pi k}{n}}$$

$$= \frac{\left(\cos\frac{2\pi k}{n}-1\right) - i\sin\frac{2\pi k}{n}}{2\left(1-\cos\frac{2\pi k}{n}\right)}$$

$$= -\frac{1}{2} - \frac{i}{2}\left(\sin\frac{2\pi k}{n} \Big/ \left(1-\cos\frac{2\pi k}{n}\right)\right)$$

So $\mathrm{Re}(z_k) = -\frac{1}{2}$ for all $k=1,\dots,n-1$, as req^{d}.

Alternatively: write $z_k = \frac{1}{\left(\exp\frac{2\pi i k}{n}-1\right)} \cdot \frac{\left(\exp\frac{-2\pi i k}{n}-1\right)}{\left(\exp\frac{-2\pi i k}{n}-1\right)}$, & proceed similarly.

Q4 a) note $u \in C^\infty(\mathbb{R}^2)$. There holds:

$u_x = \cosh x \sin y$ $\quad u_{xx} = \sinh x \sin y$

$u_y = \sinh x \cos y$ $\quad u_{yy} = -\sinh x \sin y$

$\Rightarrow \Delta u = 0$.

b) look for a harm conj v of $u \Rightarrow u, v$ satisfy C/R.

C/R$_I$: $u_x = v_y \Rightarrow v = \int \cosh x \sin y \, dy$

$= -\cosh x \cos y + \phi(x)$.

C/R$_{II}$: $u_y = -v_x \Rightarrow -\sinh x \cos y + \phi'(x) = -\sinh x \cos y$

$\Rightarrow \phi' = 0$, so $\phi = 0$ suffices.

$\Rightarrow v = -\cosh x \cos y$.

c) $u(x,y) + iv(x,y) = \sinh x \sin y - i \cosh x \cos y$.

recalling $\cosh z = \cosh x \cos y + i \sinh x \sin y$,

we see $f(z) = u + iv = -i \cosh z$

d) g is entire $\Rightarrow w$ is (also) a harm. conj of u, so $v - w = $ constant $(\in \mathbb{R})$.

(5) f is analytic on $\mathbb{C}\setminus\{1,3\}$, so it satisfies the conditions of Laurent's theorem on $\Omega_1, \Omega_2, \Omega_3$.

Note $f(z) = \frac{1}{(z-1)(z-3)} = \frac{1/2}{z-3} + \frac{1/2}{1-z}$.

Geometric series:

$$\frac{1}{1-z} = \begin{cases} \sum_{n=0}^{\infty} z^n & \text{for } |z|<1 \\ \frac{-1}{z}\frac{1}{1-1/z} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=1}^{\infty} z^{-n} & \text{for } |z|>1 \end{cases}$$

$$\frac{1}{z-3} = \begin{cases} -\frac{1}{3}\frac{1}{1-z/3} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n & \text{for } |z|<3 \\ \frac{1}{z}\frac{1}{1-3/z} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n = \sum_{n=1}^{\infty} 3^{n-1}z^{-n} & \text{for } |z|>3 \end{cases}$$

So Laurent series representations about 0 are:

Ω_1:
$$\frac{1/2}{z-3} + \frac{1/2}{1-z} = \frac{1}{2}\left(-\frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n} + \sum_{n=0}^{\infty} z^n\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(1-\frac{1}{3^{n+1}}\right)z^n$$
$$= \sum_{n=0}^{\infty}\frac{3^{n+1}-1}{2\cdot 3^{n+1}}z^n.$$

Ω_2: $$f(z) = \frac{1}{2}\left(-\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}} - \sum_{n=1}^{\infty} z^{-n}\right) = \sum_{n=0}^{\infty}\frac{-1}{2\cdot 3^{n+1}}z^n + \sum_{n=1}^{\infty}\frac{-1}{2}z^{-n}.$$

Ω_3: $$f(z) = \frac{1}{2}\left(\sum_{n=1}^{\infty}3^{n-1}z^{-n} - \sum_{n=1}^{\infty}z^{-n}\right) = \sum_{n=1}^{\infty}\frac{3^{n-1}-1}{2}z^{-n}$$

Q6 a) (i) $\exp(\frac{1}{z^2})$ is analytic on $\mathbb{C}\setminus\{0\}$, & undefined at 0, so has an isolated singularity at 0.

(ii) $f(z) = \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!} = 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \frac{1}{3!z^6} + \cdots$ ⊛

(iii) from (ii), f has an essential singularity at 0 (arbitrarily large -ve powers of z).

(iv) $\operatorname{res}_{z=0} f(z) = $ coeff of z^{-1} in ⊛ $= 0$.

b) (i) $g(z) = \frac{\cos z}{z^2 \sin z}$ ☺

so g is analytic except at the zeros of the denominator, viz. all points of the form $n\pi$, $n \in \mathbb{Z}$. Hence 0 is an isolated singularity.

(ii) from ☺, $g(z) = \frac{(1 - z^2/2! + \cdots)}{z^2(z - z^3/3! + \cdots)}$

$= \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \cdots\right)\frac{1}{z}\cdot\left(1 + \frac{z^2}{3!} + \cdots\right)$

$= \frac{1}{z^3}\left(1 - \frac{z^2}{3} + \cdots\right)$

$= \frac{1}{z^3} - \frac{1}{3z} + \cdots$ (⊛⊛)

(iii) from (ii), z has a pole of order 3 at 0 (greatest negative power).

(iv) $\operatorname{res}_{z=0} g(z) = $ coeff of z^{-1} in (⊛⊛) $= -\frac{1}{3}$.

(7)

a) f is a quotient of analytic f^{n}s: singularities are zeros of the denominator.

The zeros of $z^2 - 2z + 2$ are $\frac{2 \pm (4-8)^{1/2}}{2} = 1 \pm i$.

Near $z_1 = 1+i$, write $f(z) = \Phi(z)/z - z_1$,

where $\Phi(z) = e^{iz}/z-(1-i)$ is analytic, non zero

$\Rightarrow$ simple pole at $1+i$.

Similarly, near $z_2 = 1-i$, write $f(z) = \Psi(z)/z - z_2$,

where $\Psi(z) = e^{iz}/z-(1+i)$ is analytic, non zero

$\Rightarrow$ simple pole at $1-i$.

b) Since f has a simple pole at $1+i$,

$$\operatorname*{res}_{z=z_1} f(z) = \Phi(z_1) = e^{iz_1}/z_1-(1-i)$$

$$= \frac{e^{i(1+i)}}{(1+i)-(1-i)}$$

$$= \frac{e^{-1+i}}{2i} = -\frac{1}{2} i e^{-1+i}$$

as req^d.

c) Put $g(x) = \dfrac{\cos x}{x^2-2x+2}$, & set $J = \int_{-\infty}^{\infty} g$.

Note $x^2-2x+2 > 0$ on $\mathbb{R}$, & $|g(x)| \leq \frac{1}{x^2-2x+2}$ on $\mathbb{R}$,
so J exists (p-test).

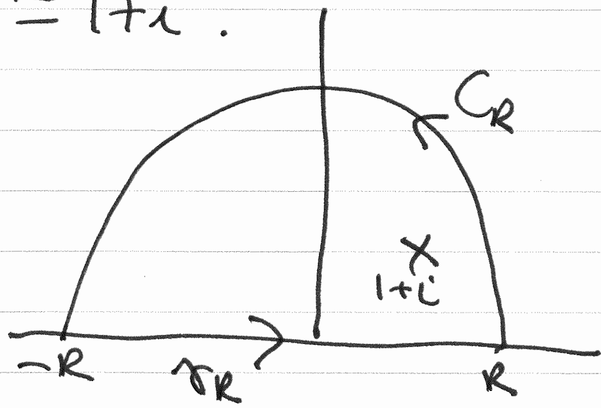
In particular, $J = PV\int_{-\infty}^{\infty} g(x)\,dx$. (+)

Consider $R > \sqrt{2}$, and a contour $C = C_R + \gamma_R$ as shown (γ_R is the line segment from $-R$ to R on the real axis: C_R is the upper semi circle centred at 0, radius R, positively oriented).

f is analytic on and inside C except for an isolated singularity at $z_1 = 1+i$.

Hence, residue theorem ⟹

$$\int_C f = 2\pi i \operatorname*{res}_{z=z_1} f \quad (*)$$



By b),

$$\text{RHS of } (*) = 2\pi i \cdot -\tfrac{1}{2} i e^{-1+i}$$

$$= \frac{\pi}{e} e^{i} = \frac{\pi}{e}(\cos 1 + i \sin 1) \quad (**)$$

$$\text{LHS of } (*) = \underset{\text{(I)}}{\int_{C_R} f} + \underset{\text{(II)}}{\int_{\gamma_R} f}$$

On C_R, put $z = x+iy$ & note $y \geq 0 \Rightarrow |e^{iz}| = e^{-y}$, with $0 < e^{-y} < 1$.

Note also on C_R, $|z^2-2z+2| > R^2-2R-2$
for R sufficiently large (e.g., $R>3$)
via the reverse Δ-inequality.
Hence, via M-L, $\left|\int_C f\right| \leq \frac{\pi R}{R^2-2R-2} \to 0$ as $R \to \infty$ (***)
(conclusion can also be drawn via Jordan's Lemma).
Note also $\lim_{R\to\infty} \mathrm{Re}\int_{\gamma_R} f = PV\int_{-\infty}^{\infty} g = \int_{-\infty}^{\infty} g$ via (+)

So, taking Re(*) and letting $R \to \infty$, via (***)
we see $J = \frac{\pi \cos 1}{e}$ as req^d.

(8) Set $h(z) = 2z^4 - \frac{1}{8}\cosh z + 1$.

Set $f(z) = 2z^4$, $g(z) = -\frac{1}{8}\cosh z + 1$

Note f & g are entire.

Set $C_1 = \{z : |z| = 1\}$.

For $z \in C_1$, $|z| = 1$, & hence $|f(z)| = 2$.

For $z \in C_1$, $|g(z)| \leq 1 + \frac{1}{8}|\cosh z|$

$$= 1 + \frac{1}{8} \cdot \left| \frac{e^z + e^{-z}}{2} \right|$$

$$< 1 + \frac{1}{16} 2e \qquad (\text{noting } |e^z| \leq e^{|z|})$$

$$< 3/2 < |f(z)|.$$

Hence by Rouché's theorem, f & $f+g = h$ have the same # of zeros (counting multiplicity) inside C_1, namely 4 (f has a quadruple zero at 0).

Set $C_2 = \{z : |z| = 1/2\}$

Set $f(z) = 1$, $g(z) = 2z^4 - \frac{1}{8}\cosh z$

On C_2, note $|\cosh z| < \frac{e^{1/2} + e^{1/2}}{2} = e^{1/2}$,

So $\quad |g(z)| < 2/16 + \sqrt{e}/8$

$$= (1 + \sqrt{e})/8 < 1 = |f(z)|.$$

Since f & g are entire, we can again apply Rouché to conclude that f & $f+g = h$ have the same # of zeros inside C_2, namely 0.

Hence h has 4 zeros counting multiplicity in $\{z : 1/2 < |z| < 1\}$.

Q 9

a) E.g. $f(z) = |z-i|^2 = x^2 + (y-1)^2 \Rightarrow u = x^2 + (y-1)^2$, $v = 0$. $u, v \in C^\infty(\mathbb{R}^2)$, C/R_I $u_x = v_y \Rightarrow 2x = 0$

C/R_{II} $u_y = -v_x \Rightarrow 2(y-1) = 0$

C/R only hold at $x = 0$, $y = 1$, i.e., at i : sufficient conditions are satisfied there $\Rightarrow$ f is diff^{ble} precisely at i.

b) Such an f would not be differentiable on any nbhd of any pt in $\mathbb{C}$, so cannot be analytic. No example.

c) None exist. If such an f existed, $g(z) = 1/f(z)$ would be non-constant, bounded & entire. No such g exists, by Liouville.

d) No such f^n exists : $T(i) = T(-i)$, & Möbius f^{n}s are 1-1.

e) e.g. $T(z) = z$.