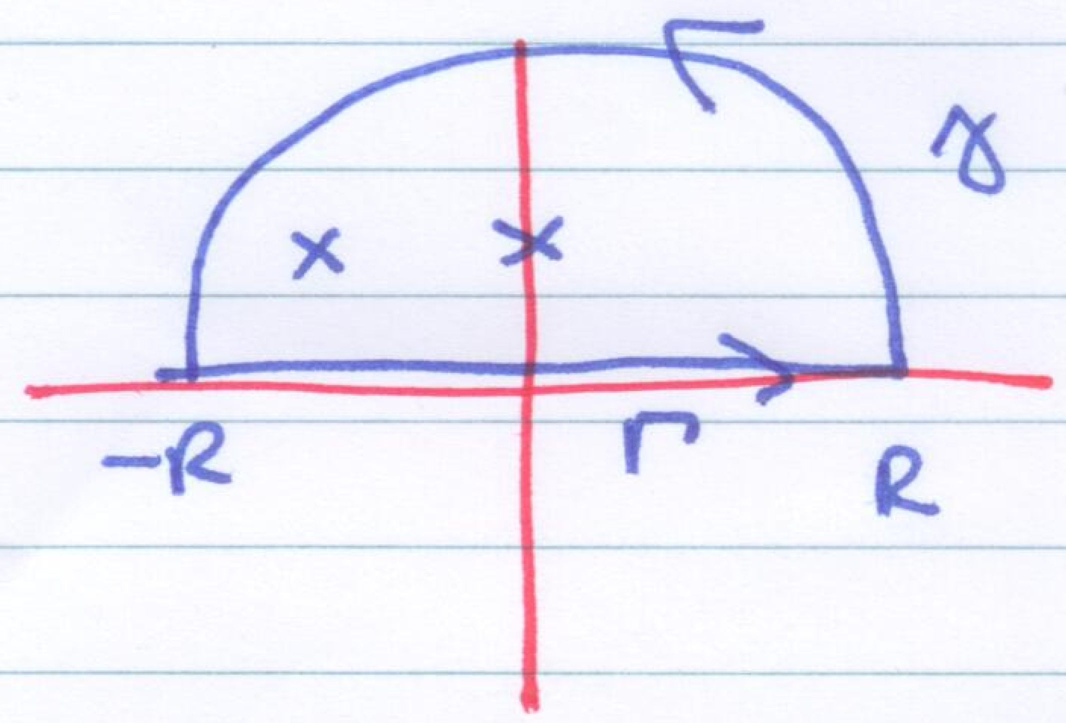
Practice qn. 4

$f(z) = \dfrac{z^2}{(z^2+1)^2(z^2+2z+2)}$

Note f is a quotient of analytic f^n's, so is analytic on $\mathbb{C} \setminus \{\text{zeros of denominator}\}$
$= \mathbb{C} \setminus \{\pm i, -1 \pm i\}$

$\int_{-\infty}^{\infty} f(x)\,dx$ exist because there are no singularities on the real axis, & $f \sim 1/x^4$ as $x \to \pm\infty$ (so via p-test, $\int_0^\infty f$ & $\int_{-\infty}^0 f$ exist, hence so does $\int_{-\infty}^{\infty} f$).

Note that because we know $\int_{-\infty}^{\infty} f$ exists, there holds: $\int_{-\infty}^{\infty} f(x)\,dx = PV\int_{-\infty}^{\infty} f(x)\,dx$. ☺



Consider $C = \Gamma + \gamma$

At i, we can write $f(z) = \dfrac{\Phi(z)}{(z-i)^2}$,

where $\Phi(z) = \dfrac{z^2}{(z+i)^2(z^2+2z+2)}$

Note Φ is analytic & nonzero at i, so f has a pole of order 2 at i, &

$$\operatorname*{res}_{z=i} f(z) = \frac{\Phi^{(2-1)}(i)}{(2-1)!} = \Phi'(i).$$

at i

$$\Phi(z) = \frac{z^2}{(z+i)^2(z^2+2z+2)}$$

$$\Phi'(z) = \frac{(z+i)^2(z^2+2z+2)(2z) - z^2\left[(z+i)^2(2z+2)+2(z+i)(z^2+2z+2)\right]}{(z+i)^4(z^2+2z+2)^2}$$

$$= \frac{(z+i)(z^2+2z+2)(2z) - z^2\left[(z+i)(2z+2)+2(z^2+2z+2)\right]}{(z+i)^3(z^2+2z+2)^2} \quad ⊛$$

$z+i\big|_{z=i} = 2i$; $z^2+2z+2\big|_{z=i} = -1+2i+2 = 1+2i$.

$$\Rightarrow ⊛ = \frac{(2i)(1+2i)(2i) + \left[2i(2i+2)+2(1+2i)\right]}{(2i)^3(1+2i)^2}$$

$$= \frac{-4-8i + -4+4i+2+4i}{(2i)^3(1+2i)^2}$$

$$= \frac{-6}{-8i(1+2i)^2} = \frac{-6}{-8i(1+4i-4)} = \frac{3}{4i(4i-3)}$$

$$= \frac{-3i}{4(3-4i)} = \frac{3i(3+4i)}{4(3-4i)(3+4i)} = \frac{-12+9i}{100}$$

At $-1+i$:

$f(z) = \psi(z)\big/(z-(-1+i))$, where

$\psi(z) = z^2\big/(z+i)^2(z-(-1-i))$

simple pole, so $\operatorname{Res}_{z=i} f(z) = \psi(-1+i) = \dfrac{(-1+i)^2}{(-1+i+i)^2(-1+i+1+i)}$

$$= \frac{(-1+i)^2}{(-1+2i)^2(2i)} = \frac{1-2i-1}{(1-4i-4)(2i)} = \frac{-1}{-3-4i}$$

$$= \frac{1}{3+4i} = \frac{3-4i}{25}$$

Applying the residue thm to the curve C given above, we see

$$\int_\gamma f + \int_\Gamma f = 2\pi i \left(\underset{z=i}{\operatorname{res}} f + \underset{z=-1+i}{\operatorname{res}} f \right)$$

$$= 2\pi i \left(\frac{9i-12}{100} + \frac{3-4i}{25} \right)$$

$$= 7\pi/50 .$$

As $R \to \infty$, ☺ $\Rightarrow \int_\Gamma f \to \int_{-\infty}^{\infty} f$.

Hence we can conclude $\int_{-\infty}^{\infty} f = \frac{7\pi}{50}$, if we can show $\lim_{R\to\infty} \int_\gamma f = 0$. ☹

note that on γ, for R large,

$$|f| = \frac{|z|^2}{|z^2+1|^2 |z^2+2z+1|}$$

$$\leq \frac{R^2}{(R^2-1)(R^2-2R-1)}$$

& $\ell(\gamma) = \pi R$.

So $$\lim_{R\to\infty} \left|\int_\gamma f\right| \leq \lim_{R\to\infty} \frac{R^2 \cdot \pi R}{(R^2-1)(R^2-2R-1)}$$

$$= \lim_{R\to\infty} \frac{\pi}{(R-1/R)(1-2/R-1/R^2)}$$

$$= 0 .$$

This shows ☹, & hence (finally)

$$\int_{-\infty}^{\infty} f = \frac{7\pi}{50}.$$