

Q1.

(a) $z^2 - 3iz - 2 = 0 \Leftrightarrow (z - 2i)(z - i) = 0$
 $\Leftrightarrow z = 2i, i$ (or use quadratic formula).

b) For $w = z^2$, we have

$w^2 - 3iw - 2 = 0$, so from (a),

$w = 2i$ (a) or

$w = i$ (b)

For (a), $z^2 = 2i \Rightarrow z^2 = 2e^{i\pi/2}$

$\Rightarrow z = \sqrt{2}e^{i\pi/4}, \sqrt{2}e^{5\pi/4}$

$= 1+i, -1-i$

For (b), $z^2 = i = e^{i\pi/2}$

$\Rightarrow z = e^{i\pi/4}, e^{5\pi/4} = \frac{1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}$

Hence, all solutions are

$\left\{ 1+i, -1-i, \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right\}$.

2a) Put $w = \tan^{-1} z$.

want to solve $z = \tan w = \frac{\sin w}{\cos w}$ for w ,

where $\cos w \neq 0$.

$$\text{So: } z = \frac{\sin w}{\cos w} = \frac{(e^{iw} - e^{-iw})/2i}{(e^{iw} + e^{-iw})/2} = \frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})}$$

$$= \frac{e^{2iw} - 1}{i(e^{2iw} + 1)}$$

$$\Leftrightarrow zi(e^{2iw} + 1) = e^{2iw} - 1$$

$$\Leftrightarrow e^{2iw}(zi - 1) = -(zi + 1) \quad (*)$$

If $zi = 1$, i.e. $z = -i$, $(*)$ means $0 = -2$, which has no solⁿ.

Similarly if $zi = -1$, i.e. $z = i$, $(*)$ means $-2e^{2iw} = 0$, which again has no solⁿ.

$$\text{Hence for } z \neq \pm i, (*) \Rightarrow e^{2iw} = \frac{-zi - 1}{zi - 1} = \frac{1 + zi}{1 - zi} = \frac{i - z}{i + z}$$

$$\Leftrightarrow 2iw = \log \frac{i - z}{i + z} \quad \text{on taking logs,}$$

$$\text{i.e. } w = \frac{1}{2i} \log \frac{i - z}{i + z} = \frac{-i}{2} \log \frac{i - z}{i + z} = \frac{i}{2} \log \frac{i + z}{i - z}$$

$$\text{b) } \tan z = 1 \Leftrightarrow z = \tan^{-1} 1 = \frac{i}{2} \log \frac{i+1}{i-1} = \frac{i}{2} \log(i)$$

$$= \frac{i}{2} [\ln|-i| + i \arg(-i)]$$

$$= \frac{i}{2} [0 + i(2k - \frac{1}{2})\pi] \quad k \in \mathbb{Z}$$

$$= (k + \frac{1}{4})\pi \quad k \in \mathbb{Z}$$

Q3.

a) (i) C/R: Since $u = x^2$, $v = y^3$, we have
 $u_x = 2x$, $v_y = 3y^2$, $v_x = u_y = 0$.

$$\text{So C/R: } \begin{cases} u_x = v_y \Rightarrow 2x = 3y^2 \\ v_x = -u_y \Rightarrow 0 = 0 \end{cases}$$

\Rightarrow C/R solved precisely on the curve $x = \frac{3}{2}y^2$.

(ii) u, v & all partials are cts on \mathbb{C} , so via (i) and "suff condⁿs" th^m, f is diff^{ble} on the curve.

$$x = \frac{3}{2}y^2.$$

(iii) f is not diff^{ble} on any nbhd of any pt in \mathbb{C} , so is nowhere analytic.

(iv) diff^{ble} \nrightarrow analytic.
 \leftarrow

$$b) (1+i)^z = \exp(z \log(1+i))$$

In order to differentiate this using the chain rule, we need to specify a single value of $\log(1+i)$, e.g. $\text{Log}(1+i)$.

Then, via the chain rule,

$$\begin{aligned} \frac{d}{dz}(1+i)^z &= \text{Log}(1+i) \exp(z \text{Log}(1+i)) \\ &= \text{Log}(1+i) (1+i)^z. \end{aligned}$$

Q4.

a) Put $f(z) = \frac{az+b}{cz+d}$

$$f(2) = 0 \Rightarrow 2a+b=0 \quad (1)$$

$$f(i) = \infty \Rightarrow ic+d=0 \quad (2)$$

$$f(0) = -2i \Rightarrow b/d = -2i \quad (3)$$

$$(3) \Rightarrow b = -2id. \text{ Sub in (1) } \Rightarrow 2a - 2id = 0$$

$$\div 2 \Rightarrow a - id = 0 \quad (4)$$

$$-i \cdot (2) \Rightarrow c - id = 0 \quad (5)$$

$$(4) \& (5) \Rightarrow a = c. \text{ Set } a = c = 1 :$$

$$(1) \Rightarrow b = -2, (2) \Rightarrow d = -i$$

$$\Rightarrow f(z) = \frac{z-2}{z-i}$$

b) No such Möbius transformation exists, as we are requiring $5i$ and i both to be mapped to i , & Möbius transformations are 1-1.