MATH3401 mid-semester sol^{n}s 2024.

Q1) $z^4 = -256 = re^{i\Theta}$ with $r = 256 = 4^4$,
$\Theta = \pi$. So $r^{1/4} = 4$

Hence, sol^{n}s are

$$\left\{ 4e^{i\pi/4},\ 4e^{i(\pi/4 + 2\pi/4)},\ 4e^{i(\pi/4+4\pi/4)},\ 4e^{i(\pi/4+6\pi/4)} \right\}$$

$$= \left\{ 4\left(\frac{1+i}{\sqrt2}\right),\ 4\left(\frac{-1+i}{\sqrt2}\right),\ 4\left(\frac{-1-i}{\sqrt2}\right),\ 4\left(\frac{1-i}{\sqrt2}\right) \right\}$$

$$= \left\{ 2\sqrt2 + 2\sqrt2 i,\ -2\sqrt2 + 2\sqrt2 i,\ -2\sqrt2 - 2\sqrt2 i,\ 2\sqrt2 - 2\sqrt2 i \right\}$$

Q2) a) $w = \coth^{-1} z \Rightarrow$

$$z = \coth w = \frac{\cosh w}{\sinh w}$$

$$= \frac{(e^{w}+e^{-w})/2}{(e^{w}-e^{-w})/2}$$

$$= \frac{e^{2w}+1}{e^{2w}-1}$$

$$\Rightarrow (e^{2w}-1)z = e^{2w}+1$$

$$\Rightarrow \quad e^{2w} = \frac{z+1}{z-1} \qquad z \neq 1$$

$$\Rightarrow \quad w = \frac{1}{2}\log\frac{z+1}{z-1} \qquad z \neq 1, -1 \quad \text{as req}^{d}.$$

b) $$\coth^{-1} i = \frac{1}{2}\log\frac{i+1}{i-1}$$

$$= \frac{1}{2}\log\left[\frac{(i+1)^2}{(i-1)(i+1)}\right]$$

$$= \frac{1}{2}\log\left(\frac{2i}{-2}\right)$$

$$= \frac{1}{2}\log(-i)$$

$$= \frac{1}{2}\left[\underbrace{\ln|-i|}_{=0} + i\arg(-i)\right]$$

$$= \frac{i}{2}\left(\frac{3\pi}{2}+2n\pi\right) \qquad n \in \mathbb{Z}$$

$$= i\left(3\pi/4 + n\pi\right) \qquad n \in \mathbb{Z}$$

Q3 a) (i) $z = x+iy \Rightarrow f(z) = x^2+y^2$

$\Rightarrow u_x = 2x, \; u_y = 2y, \quad v_x = v_y = 0$

$C/R_{I} \quad u_x = v_y \quad 2x = 0$ only on y axis

$C/R_{II} \quad u_y = -v_x \quad 2y = 0$ only on x axis

So C/R_I & C/R_{II} only hold at 0.

(Alternatively, use Wirtinger formulation of C/R:

$$f(z) = z\bar{z}, \quad \partial_{\bar{z}} f = 0 \Leftrightarrow z = 0).$$

(ii) u, u_x, u_y, v, v_x, v_y cts on $\mathbb{R}^2$. C/R hold at 0, so "suff conditions" results from Lec 15 $\Rightarrow f$ is diff^{ble} at 0.

(iii) f is not diff^{ble} on a nbhd of any pt in $\mathbb{C}$ $\Rightarrow$ nowhere analytic.

(iv) analytic $\Rightarrow$ diff^{ble}, but diff^{ble} $\not\Rightarrow$ analytic.

b) (i) No, f is unbdd on $\mathbb{C}$. Many ways of seeing this, e.g. consider $z_n = ni$

$$\Rightarrow f(z_n) = \frac{e^{-n}+e^{n}}{2} + \frac{e^{-n}-e^{n}}{2i}$$

So for n large, $f(z_n) \sim \frac{1}{2}(e^n + e^n i) = e^n\left(\frac{1+i}{2}\right)$

So $|f(z_n)| \sim \frac{e^n}{\sqrt{2}} \to \infty$ as $n \to \infty$, i.e., f is unbdd on $\mathbb{C}$.

(ii) yes. E.g., on $\mathbb{R}$, $|f(z)| \leq |\sin z| + |\cos z|$ $\quad \Delta$-ineq

$$\leq 1+1 \quad z \in \mathbb{R}$$
$$= 2.$$

(note $\mathbb{R}$ is an unbdd subset of $\mathbb{C}$).

Q4) Put $f(z) = \frac{az+b}{cz+d}$

$f(3) = 0 \Rightarrow 3a + b = 0$ ①

$f(i) = \infty \Rightarrow iz + d = 0$ ②

$f(0) = -3i \Rightarrow b/d = -3i$ ③

③ $\Rightarrow b = -3id$ Sub in ① $\Rightarrow 3a - 3id = 0$

$\div 3 \Rightarrow a - id = 0$ ④

$-i \cdot$ ② $\Rightarrow c - id = 0$ ⑤

④ - ⑤ $\Rightarrow a = c$. Set $a = c = 1$:

① $\Rightarrow b = -3$, ② $\Rightarrow d = -i$

$$\Rightarrow f(z) = \frac{z-3}{z-i}.$$

b) $T(\infty) = 0 \Leftrightarrow \lim_{z \to 0} \frac{a/z + b}{c/z + d} = 0$

$$\Leftrightarrow \lim_{z \to 0} \frac{a + bz}{c + dz} = 0$$

$$\Rightarrow a = 0, c \neq 0.$$

Since $ad - bc \neq 0$, there must also hold $b \neq 0$.