MATH3401 Mid-sem Sol'ns.

Q1. $-27000i = 27000e^{-\pi i/2}$

Since $27000^{1/3} = 30$, cube roots of $-27000i$ are

$$\left\{30e^{-i\pi/6},\ 30e^{i(-\pi/6+2\pi/3)},\ 30e^{i(-\pi/6+4\pi/3)}\right\}$$

i.e. $\left\{30\,\mathrm{cis}(-\pi/6),\ 30\,\mathrm{cis}\,\pi/2,\ 30\,\mathrm{cis}(7\pi/6)\right\}$

i.e. $\left\{30\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right),\ 30i,\ 30\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\right\}$

i.e. $\left\{15\sqrt{3}-15i,\ 30i,\ -15\sqrt{3}-15i\right\}$.

Q2 a) $w = \operatorname{cosec}^{-1} z$

$\Rightarrow \quad z = \operatorname{cosec} w = \dfrac{1}{\sin w}$

$\Rightarrow \quad \dfrac{1}{z} = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$ $\qquad z \neq 0$

$\Rightarrow e^{iw} - \dfrac{2i}{z} - e^{-iw} = 0$

$\Rightarrow (e^{iw})^2 - \dfrac{2i}{z} e^{iw} - 1 = 0$

$\Rightarrow \quad e^{iw} = \dfrac{2i/z + (-4/z^2 + 4)^{1/2}}{2}$

$\Rightarrow \quad e^{iw} = \dfrac{i}{z} + (1 - 1/z^2)^{1/2}$

$\Rightarrow \quad iw = \log\left(\dfrac{i}{z} + (1 - 1/z^2)^{1/2}\right)$

$\Rightarrow \quad w = -i \log\left(i/z + (1 - 1/z^2)^{1/2}\right)$ $\qquad z \neq 0$

b) $\operatorname{cosec} z = i \Rightarrow z = \operatorname{cosec}^{-1} i$

$= -i \log\left(i/i + (1 - 1/i^2)^{1/2}\right)$ $\qquad$ via a)

$= -i \log(1 + 2^{1/2})$

$= -i \log(1 \pm \sqrt{2})$ $\qquad$ noting $2^{1/2} = \pm\sqrt{2}$

$= \{-i(\ln(1+\sqrt{2}) + 2n\pi i),\ n \in \mathbb{Z},\ -i(\ln(\sqrt{2}-1) + (2n+1)\pi i),\ n \in \mathbb{Z}\}$

$= \{2n\pi - i\ln(1+\sqrt{2}),\ (2n+1)\pi - i\ln(\sqrt{2}-1),\quad n \in \mathbb{Z}\}$

Q3. a) (i) $u = x^2$, $v = -3y^3$

$\Rightarrow u_x = 2x$, $v_y = -9y^2$, $u_y = 0$, $v_x = 0$.

C/R_I $u_x = v_y \Rightarrow 2x = -9y^2$ only satisfied on the parabola $x = -9/2\, y^2$.

C/R_{II} $v_x = -u_y$ $0 = 0$ always true.

So C/R hold precisely on $\{x+iy : x = -9/2\, y^2\}$.

(ii) u, u_x, u_y, v, v_x, v_y cts on $\mathbb{R}^2$. C/R hold on $\{x+iy : x = -9/2\, y^2\}$, so "suff cond^{n}s" result from Lecture 15 $\Rightarrow$ f is differentiable precisely on the parabola $\{x+iy : x = -9/2\, y^2\}$.

(iii) f is not diff^{ble} on any nbhd in $\mathbb{C}$ $\Rightarrow$ nowhere analytic.

(iv) Because diff^{ble} $\not\Rightarrow$ analytic, but analytic $\Rightarrow$ diff^{ble}.

b) no, f is not bounded on $\mathbb{C}$.

Consider e.g. $z_n = n$.

$$f(z_n) = \cos z_n + \cosh z_n$$
$$= \cos n + \frac{e^{n} + e^{-n}}{2}$$
$$> \frac{e^{n}}{2} - 1 \to \infty \text{ as } n \to \infty, \quad (\text{note } \cos n \geq -1)$$

So $|f(z_n)| \to \infty$ as $n \to \infty$.

Q4. a) Put $f(z) = \dfrac{az+b}{cz+d}$

$f(3) = 0 \Rightarrow 3a + b = 0$ (1)

$f(3i) = \infty \Rightarrow 3ic + d = 0$ (2)

$f(0) = 1 \Rightarrow b/d = 1$ (3)

(3) $\Rightarrow b = d$ Sub in (1) $\Rightarrow 3a + d = 0$ (4)

(4) − (2) $\Rightarrow 3a - 3ic = 0$

ie $a = ic$.

Set $c = 1 \Rightarrow a = i$, so (4) $\Rightarrow d = -3i$,

so (3) $\Rightarrow b = -3i$.

$\Rightarrow f(z) = \dfrac{iz - 3i}{z - 3i}$.

b) No such Möb. transf exists. Möb. transfs are 1-1, but we need a map sending both $3i$ & 0 to i.