Contraction Mappings

Consider the equation

$$x = \cos x \ .$$

If we plot the graphs of $y = \cos x$ and $y = x$, we see that they intersect at a unique point for $x \sim 0.7$.

This point is called a **fixed point** of the function $\cos x$.

To determine this value more accurately, we can calculate successively

$$\begin{aligned} x_2 &= \cos(0.7) = 0.7648 \\ x_3 &= \cos(x_2) = 0.7215 \\ x_4 &= \cos(x_3) = 0.7508 \\ x_5 &= \cos(x_4) = 0.7311 \\ x_6 &= \cos(x_5) = 0.7444 \\ x_7 &= \cos(x_6) = 0.7355 \end{aligned}$$

$$x_{18} = \cos(x_{17}) = 0.7391$$

which is accurate to 4 decimal places.

Let us consider why the process of solving $x = f(x)$ by iterating $x_{n+1} = f(x_n)$ works.

Suppose for instance that we attempted to solve the equation above in its inverse form $x =$arccos(x) which has the same fixed point.

Starting from 0.7 again, we obtain

$$\begin{aligned} x_2 &= 0.7954 \\ x_3 &= 0.6511 \\ x_4 &= 0.8617 \\ x_5 &= 0.5321 \\ x_6 &= 1.0097 \end{aligned}$$

and no further terms in the sequence are defined!

The problem here is that the function $\arccos(x)$ is only defined for $-1 \leq x \leq 1$, while its range is $[0, \pi]$.

Therefore a first requirement for such an iteration to work is that, if f is defined on A, $f(A) \subset A$.

Now consider the problem of solving

$$x^2 = 2 \ .$$

If we divide by x we obtain the iterative form

$$x = f(x) = \frac{2}{x}$$

If we consider positive real values, we see that $f((0,\infty)) = (0,\infty)$ or indeed $f([1,2]) = [1,2]$.

Therefore this function has the property that $f(A) \subset A$.

However, when we iterate starting with $x_1 = a$, we obtain the sequence

$$a,\ \frac{2}{a},\ a,\ \frac{2}{a},\ a, \ldots$$

which does not converge to the fixed point $\sqrt{2}$.

Therefore we need some further criterion.

If we want our sequence to converge to the fixed point $a = f(a)$, then we expect that

$$\begin{aligned} |x_{n+1} - a| &< |x_n - a| \\ |f(x_n) - f(a)| &< |x_n - a| \end{aligned}$$

However, since the value a is in general unknown, we consider instead the requirement

$$|f(x) - f(y)| < |x - y|$$

for all $x \neq y \in A$.

(Note in passing that this condition means that f is uniformly continuous on A.)

In the case of $f(x) = \cos x$, we have

$$\begin{aligned} |\cos x - \cos y| = &\left| -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right| \\ =&2\left|\sin\left(\frac{x+y}{2}\right)\right|\left|\sin\left(\frac{x-y}{2}\right)\right| \\ <&2 \times 1 \times \frac{|x-y|}{2} = |x-y| \end{aligned}$$

while for $f(x) = \frac{2}{x}$,

$$|f(1.1) - f(1)| = \left|\frac{20}{11} - 2\right| = \frac{2}{11} > \frac{1}{10}$$

(We can obtain a convergent iteration for the solution of $x^2 = 2$ by adding x to both sides, and then dividing by $x + 1$.

$$\begin{gathered} x^2 + x = x + 2\ ;\ x = \frac{x+2}{x+1} \\ f(x) = \frac{x+2}{x+1} = 1 + \frac{1}{x+1} = 2 - \frac{x}{x+1} \\ f([0,\infty)) = (1,2] \subset [0,\infty) \\ |f(x) - f(y)| = \left|\frac{1}{x+1} - \frac{1}{y+1}\right| = \frac{|x-y|}{(1+x)(1+y)} < |x-y| \end{gathered}$$

for all $x \neq y \in [0,\infty)$ in this case.

The iteration

$$x_1 = 1\ ;\ x_{n+1} = \frac{x_n + 2}{x_n + 1}$$

gives the sequence of approximations to $\sqrt{2}$ which we considered at the start of this course.

However, if we consider $f : [1, \infty) \to [1, \infty)$ given by

$$f(x) = x + \frac{1}{x}$$

$$\left| x + \frac{1}{x} - y - \frac{1}{y} \right| = |x - y| \left| 1 - \frac{1}{xy} \right| < |x - y|$$

but the equation $x = f(x)$ has no solution.

This difficulty is overcome by strengthening the condition slightly.

Definition: The function f from a metric space (A, d) to (A, d) is a **contraction mapping** on A if there is a constant k, $(0 \leq)k < 1$ such that

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in A$.

We note again that a contraction mapping on A is uniformly continuous on A.

The final problem which we have to consider has already been addressed at the start of the course.

The iteration

$$x_1 = 1 \ ; \ x_{n+1} = \frac{x_n + 2}{x_n + 1}$$

is a contraction mapping on $\mathbb{Q}^+$, but does not converge because $\sqrt{2}$ is not rational.

We therefore need to include completeness in our list of requirements.

Banach's Fixed Point Theorem.

If f is a contraction mapping on the complete metric space (A, d), then the equation $x = f(x)$ has a unique solution in A, and for any initial value $x_0 \in A$, the sequence

$$x_{n+1} = f(x_n)$$

converges to this fixed point of the function.

Consider any such sequence.

$$\begin{aligned} d(x_{n+1}, x_n) =& d(f(x_n), f(x_{n-1})) \\ \leq & \ kd(x_n, x_{n-1}) \\ \leq & \ k^2 d(x_{n-1}, x_{n-2}) \\ & \dots \\ \leq & \ k^n d(x_1, x_0) \end{aligned}$$

Therefore, for $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \\ &\leq \sum_{i=n}^{m-1} k^i d(x_1, x_0) \\ &= \frac{k^n - k^m}{1 - k} d(x_1, x_0) \\ &< k^n \frac{d(x_1, x_0)}{1 - k} \end{aligned}$$

Since $0 \le k < 1$, $k^n \to 0$ as $n \to \infty$.
Therefore, for any $\epsilon > 0$, we can find an integer N such that

$$k^n \frac{d(x_1, x_0)}{1-k} < \epsilon \ \forall \ n > N \ .$$

But then, $d(x_m, x_n) < \epsilon$ for all $m > n > N$, and $\{x_n\}$ is a Cauchy sequence in A.
Since A is complete, this sequence has a limit a in A.
Furthermore, since f is continuous on A, the sequence $\{f(x_n)\}$ converges to $f(a)$. But $\{f(x_n)\} = \{x_{n+1}\}$ converges to a, so that a is a fixed point of the function in A.

Suppose that b is another fixed point of f in A: $b = f(b)$.
Then

$$d(a, b) = d(f(a), f(b)) \le kd(a, b)$$

which implies that $d(a, b) = 0$ and hence that $a = b$.
Therefore the fixed point of the function in A is unique.

Note that if we take the limit as $m \to \infty$ in the estimate for $d(x_m, x_n)$ we obtain

$$d(x_n, a) \le k^n \frac{d(x_1, x_0)}{1-k} \ .$$

Taking $n = 0$, and remembering that the choice of $x_0 \in A$ is arbitrary, we obtain the error estimate

$$d(x, a) \le \frac{1}{1-k} d(x, f(x)) \ .$$

Note that this result asserts both **existence** and **uniqueness**.

Note also that as a numerical procedure the iteration process is stable.
Since the sequence converges to the fixed point for any starting value, any errors in the calculations are damped out.

e.g. Consider $f(x) = \frac{1}{7}(x^3 + x^2 + 1)$.
For $0 \le x \le 1$, $1 \le 1 + x^2 + x^3 \le 3$, so that $f([0, 1]) = [\frac{1}{7}, \frac{3}{7}] \subset [0, 1]$.

$$\begin{aligned} |f(x) - f(y)| &= \frac{1}{7}|x^3 - y^3 + x^2 - y^2| \\ &= \frac{1}{7}|x - y||x^2 + xy + y^2 + x + y| \\ &\le \frac{5}{7}|x - y| \end{aligned}$$

for $x, y \in [0, 1]$.
Therefore f is a contraction mapping on $[0, 1]$, which is complete, so that there is a unique solution of

$$\begin{gathered} x = \frac{1}{7}(x^3 + x^2 + 1) \\ x^3 + x^2 - 7x + 1 = 0 \end{gathered}$$

in $[0, 1]$, and we can determine this solution numerically by iterating the function (from any starting value in $[0, 1]$).

e.g. for $x_0 = 1$, we have

$$\begin{aligned} x_1 &= 0.4286 \\ x_2 &= 0.1749 \\ x_3 &= 0.1480 \\ x_4 &= 0.1464 \\ x_5 &= 0.1464 \end{aligned}$$

Note that the cubic equation

$$x^3 + x^2 - 7x + 1 = 0$$

has three real roots.

If we wish to calculate the other two in this way, we will need to construct two other contraction mappings on appropriate intervals.

This process can be facilitated by the following result:

Let $f : [a, b] \to [a, b]$ be differentiable.

Then f is a contraction on $[a, b]$ if and only if there exits a number $k < 1$ such that

$$|f'(x)| \leq k \ \forall \ x \in (a, b)$$

.

If f is a contraction on $[a, b]$, then there is a $k; (0 \leq) k < 1$ such that for any x, $x + \delta x$ in (a, b),

$$\begin{aligned} |f(x + \delta x) - f(x)| &\leq k|\delta x| \\ \left| \frac{f(x + \delta x) - f(x)}{\delta x} \right| &\leq k \end{aligned}$$

Taking the limit as $\delta x \to 0$, and noting that the modulus function is continuous, we obtain

$$|f'(x)| \leq k$$

Conversely, if f is differentiable on $[a, b]$, then by the mean value theorem, for any $x \neq y$ in $[a, b]$ there is a point c between x and y such that

$$f(x) - f(y) = f'(c)(x - y) \ ,$$

and if $|f'(x)| \leq k < 1$ on (a, b)

$$|f(x) - f(y)| \leq k|x - y| \ .$$

e.g.

If $f(x) = \cos x$, since the maximum modulus of the derivative is 1 on $\mathbb{R}$, this is not a contraction mapping on $\mathbb{R}$.

However, on $[0, 1]$, $|f'(x)| \leq \sin 1 = 0.8415$, so that we have a contraction on $[0, 1]$.

If $f(x) = x^3 + x^2 + 1$, then $f'(x) = 3x^2 + 2x \leq 5$ on $[0, 1]$. Therefore for any $k > 5$, $\frac{1}{k}(x^3 + x^2 + 1)$ will be a contraction on $[0, 1]$, and the iteration will converge to a solution of $x^3 + x^2 - kx + 1 = 0$.

Suppose that we want to solve $F(x) = 0$, where F is differentiable.
We can write this as a fixed point problem by setting

$$f(x) = x + cF(x)$$

for some constant c.

If we know an approximate value a for the desired solution, we can ensure that we have a contraction is some neighbourhood of a by making $f'(a) = 0$ so that $|f'|$ is locally small;

$$f'(x) = 1 + cF'(x) \ ;$$
$$f'(a) = 1 + cF'(a) = 0 \text{ if } c = -1/F'(a)$$

giving

$$f(x) = x - \frac{F(x)}{F'(a)}$$

For example, the equation $F(x) = x^3 + x^2 - 7x + 1 = 0$ has a solution near $x = 2$.
Since $F'(x) = 3x^2 + 2x - 7$, $F'(2) = 9$ and we can try

$$f(x) = \frac{1}{9}(16x - x^3 - x^2 - 1)$$

.

With $x_0 = 2$,

$$\begin{aligned} x_1 &= 2.1111 \\ x_2 &= 2.1014 \\ x_3 &= 2.1030 \\ x_4 &= 2.1027 \\ x_5 &= 2.1028 \end{aligned}$$

The remaining root is near $x = -3$.
Students should determine a contraction mapping which will determine it.

At the expense of more complication, we can update the derivative at each step.
The resulting mapping

$$f(x) = x - \frac{F(x)}{F'(x)}$$

is known as "Newton's Method".