Suppose that we have a function $f : \mathbb{R} \to \mathbb{R}$.

Let $a \in \mathbb{R}$.

We say that $f(x)$ tends to the limit l as x tends to a;

$$\lim_{x \to a} f(x) = l \ ;$$

if, given any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x) - l| < \epsilon \ \forall \ 0 < |x - a| < \delta \ .$$

Note:

1. The function need not be defined at $x = a$ in order to have a limit there. This is typically the case when we are evaluating derivatives.

2. Even if $f(a)$ exists, it is not necessary that $f(a) = l$.

e.g. Consider the function $f(x)$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1 + x^2)^n} \ .$$

$f(0) = 0$, but for $x \neq 0$, we can sum the geometric progression to obtain $f(x) = 1 + x^2$. Therefore

$$\lim_{x \to 0} f(x) = 1 \ .$$

3. For given ϵ, the value of δ is not fixed. If the property holds for some $\delta_1 > 0$ then it holds for any $0 < \delta \leq \delta_1$.

If $f(a)$ exists **and** $f(a) = l$, we say that $f(x)$ is **continuous** at $x = a$.

That is:

$f(x)$ is continuous at $x = a$, if $f(a)$ exists, and, given any $\epsilon > 0$, there is a $\delta > 0$, (which value depends usually on both x and ϵ) such that

$$|f(x) - f(a)| < \epsilon \ \forall \ |x - a| < \delta \ .$$

Note that since the first inequality is trivially satisfied when $x = a$, there is no need to exclude this case from the definition.

This definition of continuity does not accord with the intuitive idea of "drawing the graph without taking the pencil off the paper".

This is because it defines continuity at a point. This allows some unexpected results.

For example, consider the following functions:

1. $f(x) = x$ if x is rational, $f(x) = -x$ if x is irrational.

This function is continuous only at the point $x = 0$.

2. $f(x) = \frac{1}{q}$ when $x = \frac{p}{q}$, p and $q(> 0)$ coprime;
$f(x) = 0$ when x is irrational.

If $a = \frac{p}{q}$ is rational, then if we take $\epsilon = \frac{1}{2q}$, for every irrational x,

$$|f(x) - f(a)| = \left|0 - \frac{1}{q}\right| = \frac{1}{q} > \epsilon .$$

Since every interval $|x - a| < \delta$ contains irrational points, there is no $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \ \forall \ |x - a| < \delta .$$

Therefore $f(x)$ is discontinuous at every rational value of x.

On the other hand, given any $\epsilon > 0$, we can find an integer Q such that $\epsilon > \frac{1}{Q}$.
For any irrational value a, consider the interval $|x - a| < \frac{1}{2}$.
In this interval there are at most q rational numbers of the form $\frac{p}{q}$, so that there are at most $Q(Q-1)/2$ such numbers for which $\frac{1}{q} > \frac{1}{Q}$.
If we set

$$\delta = \min_{q < Q} \left|\frac{p}{q} - a\right|$$

then $\delta \neq 0$ since a is irrational, and if $|x - a| < \min(\frac{1}{2}, \delta)$,

$$|f(x) - f(a)| = |0 - 0| < \epsilon$$

when x is irrational;
while if $x = \frac{p}{q}$, $q \geq Q$, so that

$$|f(x) - f(a)| = \frac{1}{q} \leq \frac{1}{Q} < \epsilon .$$

Therefore this function is continuous for every irrational value of x.

This definition of continuity allows us to extend our consideration of the convergence of sequences.
In particular, we have the following result:
The function $f(x)$ is continuous at $x = a$ if and only if for every sequence $\{x_n\}$ which converges to a, the sequence $\{f(x_n)\}$ converges to $f(a)$.
This result is of central importance to analysis.

The proof of the first part is straightforward.
If the function $f(x)$ is continuous at $x = a$, then given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \ \forall \ |x - a| < \delta .$$

For any sequence which converges to a, for this value δ we can find an integer N such that

$$|x_n - a| < \delta \ \forall \ n > N .$$

Therefore

$$|f(x_n) - f(a)| < \epsilon \ \forall \ n > N .$$

i.e. $\{f(x_n)\}$ converges to $f(a)$.

To prove the converse result that $f(x_n) \rightarrow f(a)$ for every sequence $x_n \rightarrow a$ implies that $f(x)$ is continuous at $x = a$, we show that if $f(x)$ is NOT continuous at $x = a$ then we can construct a sequence $\{x_n\}$ which converges to a but for which $\{f(x_n)\}$ does not converge to $f(a)$.

Note that if $f(a)$ does not exist, the result is trivially true.

Otherwise, if $f(x)$ is not continuous at $x = a$, then there is some value $\epsilon_1 > 0$ such that for every $\delta > 0$ we can find a number X such that $|X - a| < \delta$ and $|f(X) - f(a)| \geq \epsilon_1$.

Note that $X \neq a$ so that $|X - a| > 0$.

Begin with $\delta_1 = 1$. There is a value X_1 such that

$$|X_1 - a| < \delta_1 = 1 \; ; \; |f(X_1) - f(a)| > \epsilon_1 \; .$$

Choose $\delta_2 = \frac{1}{2}|X_1 - a| > 0$. There is a value X_2 such that

$$|X_2 - a| < \delta_2 < \frac{1}{2} \; ; |f(X_2) - f(a)| > \epsilon_1 \; .$$

We proceed inductively defining $\delta_{n+1} = \frac{1}{2}|X_n - a|$ and then determining the value X_{n+1}.

The sequence $\{X_n\}$ has the property that

$$|X_n - a| < \frac{1}{2^{n-1}} \; \forall \; n \; ,$$

so that it converges to a. However,

$$|f(X_n) - f(a)| > \epsilon_1 \; \forall n \; ,$$

so that this sequence does not converge to $f(a)$.

As an illustration, consider the second function discussed above.

If we take $a = \pi$, and consider the sequence

$$\{3, \; 3.1, \; 3.14, \; 3.141, \; 3.1415, \; 3.14159, \; \ldots \}$$

then the sequence $\{f(x_n)\}$ is

$$\{1, \; 0.1, \; 0.02, \; 0.001, \; 0.0005, \; 0.00001, \; \ldots \}$$

which converges to $f(\pi) = 0$.

On the other hand, for $a = \frac{1}{3}$, we can consider the sequence

$$\{0.3, \; 0.33, \; 0.333, \; \ldots \}$$

which converges to a.

The sequence $\{f(x_n)\}$ is

$$\{0.1, \; 0.01, \; 0.001, \; \ldots \}$$

which converges to 0, while $f(a) = \frac{1}{3}$.

This is in accord with the fact that $f(x)$ is continuous for x irrational, but discontinous for x rational.

We could in fact use this property to give another proof that this function is not continuous at any rational point.

For any rational number r, consider the sequence

$$x_n = r + \frac{\sqrt{2}}{n}$$

which converges to r.

Each x_n is irrational, so that $f(x_n) = 0$ for each x_n, and the sequence $\{f(x_n)\}$ does not converge to $f(r)$. Therefore f is not continuous at r.

As a second illustration, consider the sequence $\{p_n^2/q_n^2\}$ which we showed converges to 2.

If the function $f(x)$ is continuous at $x = 2$, then

$$f\left(\frac{p_n^2}{q_n^2}\right) \to f(2) \ .$$

In particular, for $f(x) = \sqrt{x}$, we obtain the result

$$\frac{p_n}{q_n} \to \sqrt{2} \ .$$

We recover our intuitive idea of continuity by considering **continuity on an interval.**

Given some interval I, the function $f(x)$ is continuous on I if it is continuous (in the pointwise sense above) at every point of I.

In particular, we have the **Intermediate Value Theorem**:

If the function f is continuous on the interval I, then for every $a < b$ in I, and for every l between $f(a)$ and $f(b)$, there is a number c in (a, b) such that $f(c) = l$.

(The converse is not true; a function may have the intermediate value property without being continuous. Consider for example $f(x) = x - 1/x$ with $a < -1$ and $b > 1$.)

Suppose without loss of generality that $f(a) < f(b)$, and consider the set of all numbers $x \in [a, b]$ for which $f(x) < l$.

Since a is in this set, it is non-empty.

This set is bounded above by b, therefore it has a least upper bound c. We will show that $f(c) = l$.

Note that since $f(b) > l$, and f is continuous at b, there is a $\delta > 0$ such that

$$|f(x) - f(b)| < f(b) - l$$
$$l < f(x) < 2f(b) - l$$

for all $|x - b| < \delta$; $b - \delta < x < b + \delta$, so that $c \leq b - \delta$.

Similarly, $c \geq a + \delta$ for some $\delta > 0$.

Suppose that $f(c) < l$. We know that $f(x)$ is continuous at $x = c$. Therefore, for $\epsilon = l - f(c) > 0$, we can find $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \ \forall \ |x - c| < \delta \ .$$

Therefore
$$f(x) < l \; \forall \; x < c + \delta$$
and c is not an upper bound for the set.

On the other hand, if $f(c) > l$, we choose $\epsilon = f(c) - l > 0$.

There is a $\delta > 0$ such that
$$|f(x) - f(c)| < f(c) - l \; \forall \; |x - c| < \delta$$
$$f(x) < l \; \forall \; x > c - \delta$$
so that c is not the least upper bound.

Together these two contradictions show that $f(c) = l$ as required.

Uniform continuity.

Suppose that the function f is continuous on an interval I.

For any given $\epsilon > 0$ and for each $x \in I$, we can find some $\delta > 0$ such that
$$|f(y) - f(x)| < \epsilon \; \forall \; \|y - x\| < \delta$$
Therefore the set of values of δ for which this is true is non-empty.

If the set of values is not bounded above (which happens if the function is a constant) choose $\delta(x, \epsilon) = 1$.

Otherwise, let $\delta(x, \epsilon)$ be the supremum of this set of values.

Note that for any $\delta < \delta(x, \epsilon)$,
$$|f(y) - f(x)| < \epsilon \; \forall \; \|y - x\| < \delta$$
Now consider the function $\delta(x, \epsilon)$ for $x \in I$.

It is defined for every $x \in I$, so that the set of values is non-empty.

Also, each $\delta(x, \epsilon) > 0$, so that this set of values is bounded below.

Therefore, this set has a greatest lower bound, $\delta(I, \epsilon)$ say.

If $\delta(I, \epsilon) > 0$ for every $\epsilon > 0$, we say that the function is uniformly continuous on the interval.

This means that given $\epsilon > 0$, we can find $\delta > 0$ (which depends on ϵ and on the interval I but not on particular points in I) such that
$$|f(y) - f(x) < \epsilon$$
for every $x, y \in I$ such that
$$|y - x| < \delta$$
For example, consider $f(x) = \sin x$.
$$\sin y - \sin x = 2 \sin \left(\frac{1}{2}(y - x) \right) \cos \left(\frac{1}{2}(y + x) \right)$$
For all values of θ,
$$|\sin \theta| \le |\theta|$$
$$|\cos \theta| \le 1$$
so that
$$|\sin y - \sin x| \le 2 \left| \frac{1}{2}(y - x) \right| = |y - x|$$
Therefore, given any $\epsilon > 0$,
$$|\sin y - \sin x| < \epsilon \; \forall \; |y - x| < \epsilon$$
and this function is uniformly continuous on $\mathbb{R}$.

On the other hand, consider the function $f(x) = x^2$.
For $x \in \mathbb{R}$,
$$\delta(x, \epsilon) = \sqrt{x^2 + \epsilon} - |x| = \frac{\epsilon}{\sqrt{x^2 + \epsilon} + |x|} \ .$$

While this is positive for all $x \in \mathbb{R}$, so that f is continuous on $\mathbb{R}$, $\delta(x, \epsilon) \to 0$ as $|x| \to \infty$, so that f is not uniformly continuous on $\mathbb{R}$.

On the other hand, if we restrict the function to a finite interval $[-a, a]$,

$$\delta(x, \epsilon) \geq \sqrt{a^2 + \epsilon} - a$$

for all $x \in [-a, a]$, so that f is uniformly continuous on the finite interval.

Finally, consider the function $f(x) = 1/x$ on the interval $(0, 1]$.
In this case
$$\delta(x, \epsilon) = \frac{\epsilon x^2}{1 + \epsilon x}$$

which goes to 0 as $x \to 0+$.

Therefore f is continuous on $(0, 1]$ but not uniformly continuous.

Uniform continuity is relevant to Cauchy sequences.

If $\{x_n\}$ is a Cauchy sequence in some set S, and f is uniformly continuous on S, then $\{f(x_n)\}$ is also a Cauchy sequence.

Given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \ \forall \ x, y \in S \ ; \ |x - y| < \delta \ .$$

Since $\{x_n\}$ is Cauchy, given this $\delta > 0$ we can find an integer N such that

$$|x_n - x_m| < \delta \ \forall \ m, n > N \ .$$

But then
$$|f(x_n) - f(x_m)| < \epsilon \ \forall \ m, n > N \ ,$$

and the sequence $\{f(x_n)\}$ is Cauchy.

Therefore, for any Cauchy sequence $\{x_n\}$ in $\mathbb{R}$, $\{\sin(x_n)\}$ is also a Cauchy sequence, and since Cauchy sequences are bounded, $\{x_n^2\}$ is also Cauchy.

On the other hand, $\{\frac{1}{n}\}$ is a Cauchy sequence on $(0, 1]$, but $\{n\}$ is not a Cauchy sequence.