Homeomorphisms.

A homeomorphism between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a one - one correspondence such that f and f^{-1} are both continuous.

Consequently, for every $U \in \mathcal{T}_X$ there is $V \in \mathcal{T}_Y$ such that $V = F(U)$ and $U = f^{-1}(V)$.

Furthermore, since

$$f(U_1 \cap U_2) = f(U_1) \cap f(U_2)$$

and

$$f(U_1 \cup U_2) = f(U_1) \cup f(U_2)$$

this equivalence extends to the structures of the spaces.

Example

Any open interval in $\mathbb{R}$ (with the inherited topology) is homeomorphic to $\mathbb{R}$.

One possible function from $\mathbb{R}$ to $(-1, 1)$ is $f(x) = \tanh x$, and by suitable scaling this provides a homeomorphism onto any finite open interval (a, b).

$$f(x) = \frac{b+a}{2} + \frac{b-a}{2} \tanh x$$

For semi-infinite intervals we can use with $f(x) = a + e^x$ from $\mathbb{R}$ to (a, ∞) and $f(x) = b - e^{-x}$ from $\mathbb{R}$ to $(-\infty, b)$.

Since homeomorphism is an equivalence relation, this shows that all open intervals in $\mathbb{R}$ are homeomorphic.

nb The functions chosen above are not unique.

On the other hand, no closed interval $[a, b]$ is homeomorphic to $\mathbb{R}$, since such an interval is compact in R, and hence $f([a, b])$ is compact for any continuous function f.

Example 2

The function $f(z) = z^2$ induces a homeomorphism between the complex plane $\mathbb{C}$ and a two sheeted Riemann surface, which we can construct piecewise using f.

Consider first the half-plane $x > 0 \subset \mathbb{C}$.

The function f maps this half plane onto a complex plane missing the negative real axis.

The first quadrant maps onto the upper half plane, and the fourth quadrant onto the lower half plane.

Now take the half-plane $y > 0 \subset \mathbb{C}$.

This half plane maps onto a complex plane missing the positive real axis.

We identify the upper half of this cut plane with the upper half of the first cut plane, since they both represent the map of the first quadrant.

However the lower half of this second cut plane is the map of the second quadrant, and as such is distinct from the lower half of the first cut plane.

Continuing, we take the half plane $x < 0 \subset \mathbb{C}$.

This half plane maps onto a complex plane missing the negative real axis, similar to the first one.

However, in this case the upper half plane is the map of the third quadrant, while the lower half plane is the map of the second quadrant, and as such is identified with the lower half plane of the second cut plane.

Finally, we consider the half plane $y < 0 \subset \mathbb{C}$.

This half plane maps onto a complex plane missing the positive real axis.

The upper half plane is the map of the third quadrant, and is identified with the upper half of the third cut plane.

Lastly, the lower half plane is the map of the fourth quadrant, and is identified with the lower half of the first cut plane.

The continuous surface defined by these submaps is the Riemann surface of the function.

In general, any function f regular on a domain $\mathcal{D} \subset \mathbb{C}$, defines a homeomorphism from $\mathcal{D}$ to $f(\mathcal{D})$, where the surface $f(\mathcal{D})$ is built up by overlaying the maps of neighbourhoods $\mathcal{N}(z_0, \epsilon) \subset \mathcal{D}$ in which $f' \neq 0$.

The points at which $f' = 0$ determine branch points in $f(\mathcal{D})$.

Graphs.

If we consider the function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$, it does not induce a homeomorphism since it is not one - one.

On the other hand, if we sketch the graph of the function in $\mathbb{R}^2$, there is an obvious homeomorphism from $\mathbb{R}$ to the graph;

$$x \to (x, x^2) \ .$$

We can extend this concept to a general function F continuous from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$.

The graph of F is the subset

$$\mathcal{G}_F = \{(x, F(x)); x \in X\} \subset X \times Y$$

Then $\mathcal{G}_F$ with the topology inherited from the product topology is homeomorphic to $(X, \mathcal{T}_X)$.

The function f from X to $\mathcal{G}_F$ given by

$$f(x) = (x, F(x))$$

is continuous if and only if its components are.

$p_1 \circ f$ is the identity map on X and hence continuous;

$p_2 \circ f$ is F is continuous by definition;

Therefore f is continuous from X to $\mathcal{G}_F$.

The inverse f^{-1} is p_1, the projection from $X \times Y$ to X, which is continuous.

Closed sets.

Given a topological space $(X, \mathcal{T})$, we say that the set V is closed (in $(X, \mathcal{T})$) if $X \setminus V \in \mathcal{T}$.

Hence ϕ and X are closed, the union of two closed sets is closed, and the intersection of any number of closed sets is closed.

If $(Y, \mathcal{T}')$ is a subspace of $(X, \mathcal{T})$, then $V \subset Y$ is closed in $(Y, \mathcal{T}')$ if and only if $V = Y \cap W$ for some W closed in $(X, \mathcal{T})$.

Suppose V is closed in $(Y, \mathcal{T}')$.

Then $V = Y \setminus U'$ for some $U' \in \mathcal{T}'$.

By the definition of the subspace topology, $U' = Y \cap U$ for some $U \in \mathcal{T}$, so that

$$V = Y \setminus (Y \cap U) = (Y \cap X) \setminus (Y \cap U) = Y \cap (X \setminus U) = Y \cap W$$

where W is closed in $(X, \mathcal{T})$.

Conversely, if $V = Y \cap W$ where W is closed in $(X, \mathcal{T})$, then

$$Y \setminus V = Y \setminus (Y \cap W) = (Y \cap X) \setminus (Y \cap W) = Y \cap (X \setminus W)$$

is open in $(Y, \mathcal{T}')$

It follows that if Y itself is closed in $(X, \mathcal{T})$, then $V \subset Y$ is closed in $(Y, \mathcal{T}')$ if and only if V is closed in $(X, \mathcal{T})$.

If V is closed in $(Y, \mathcal{T}')$, then $V = Y \cap W$ where W is closed in $(X, \mathcal{T})$. Since the intersection of two closed sets is closed, V is closed in $(X, \mathcal{T})$

Conversely, if $V \subset Y$ is closed in $(X, \mathcal{T})$, $V = Y \cap V$ is closed in $(Y, \mathcal{T})$.

In view of the similarities of the results for open and closed sets, the structure of the topological space could be stated in terms of "closed sets" instead of "open sets".

This is further highlighted by the result

The function f from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$ is continuous if and only if, for every V closed in Y, $f^{-1}(V)$ is closed in X.

Limit points.

Let $(X, \mathcal{T})$ be a topological space, and $H \subset X$.

A point $x \in X$ is a limit point of H if every open set containing x contains a point of H other then x; that is,

$x \in T \in \mathcal{T}$ implies $(T \setminus \{x\}) \cup H \neq \phi$.

When the topology is generated by a metric, we have seen that this implies that every open set containing x contains an infinite number of points of H.

However, if the topology is not so generated this is not necessary.

For example consider the MATH 3402 class with the topology generated by the subsets $S_{11}, S_{12}, S_{21}, S_{22}$.

Let H be the subset of students whose surnames begin with H.

(AFAIK this involves one student in S_{11} and one in S_{12}.)

Then, every element of S_{11} and of S_{12}, with the exception of those in H itself, is a limit point of H.

If however there are two boys (say) whose surnames begin with H, they would both be limit points.

The **closure** $Cl(H)$ of a subset H in $(X, \mathcal{T})$, is the union of H with all its limit points in $(X, \mathcal{T})$.

In the example above, $Cl(H) = S_{11} \cup S_{12} = A_1$.

Hence $x \in Cl(H)$ if and only if every open set containing x contains a point of H.

A subset H is closed in $(X, \mathcal{T})$ if and only if $H = Cl(H)$.

By definition $H \subset Cl(H)$.

If H is closed, and $x \in (X \setminus H)$, then $X \setminus H$ is an open set containing x. Therefore x is not a limit point of H, and $Cl(H) \subset H$. Hence $H = Cl(H)$.

Conversely, if $H = Cl(H)$, then for $x \in (X \setminus H)$, there is some $U_x \in \mathcal{T}$ such that $x \in U_x$ and $U_x \cap H = \phi$.

Therefore $U_x \subset (X \setminus H)$.

Since $X \setminus H \subset \cup_x U_x$,

$$X \setminus H = \bigcup_x U_x \in \mathcal{T}$$

and H is closed.

The closure of H is closed.

If $x \in Cl(Cl(H))$, then any open set U containing x contains a point $y \in Cl(H)$. Therefore $U \cap H \neq \phi$, and $x \in Cl(H)$.

Therefore $Cl(Cl(H)) \subset Cl(H)$, and since $Cl(H) \subset Cl(Cl(H))$, these sets are equal, and $Cl(H)$ is closed.

To continue the above example, we have

$$Cl(H) = A_1 = X \setminus A_2$$

which is closed in $(X, \mathcal{T}_3)$

A subset H is **dense** in $(X, \mathcal{T})$ if $Cl(H) = X$.

The most obvious example of this is $\mathbb{Q}$ as a subset of $\mathbb{R}$, and we will meet other similar examples later.

If we consider the subset T in MATH 3402 of students whose surnames begin with T, then $H \cup T$ is dense in $(X, \mathcal{T}_3)$.

Also, H itself is dense in $(X, \mathcal{T}_2)$.

Interior.

The interior $int(H)$ of a subset H is the union of all open sets contained in H.

For example, with MATH 3402 and $\mathcal{T}_3$, $int(H) = \phi$.

Less trivially, the interior of $\mathbb{Q}$ considered as a subset of $\mathbb{R}$ is also empty.

A subset H is **nowhere dense** in $(X, \mathcal{T})$ if $int(Cl(H)) = \phi$

For example, $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

This definition is equivalent to the statement that H is nowhere dense in $(X, \mathcal{T})$ if $X \setminus Cl(H)$ is dense in $(X, \mathcal{T})$.

$int(Cl(H)) = \phi$ if and only if $U \cap (X \setminus Cl(H)) \neq \phi$ for every $U \in \mathcal{T}$ which means $X \setminus Cl(H)$ is dense in $(X, \mathcal{T})$.

Hence, if H is closed, H is nowhere dense if $X \setminus H$ is dense.

Finally, the **boundary** $b(H)$ of a subset H is

$$b(H) = Cl(H) \cap Cl(X \setminus H)\ .$$

Concluding our MATH 3402 example, we have

$$\begin{gathered} Cl(H) = A_1 \\ Cl(X \setminus H) = X \\ \text{therefore} \;\; b(H) = A_1 \end{gathered}$$