Topological Linear Spaces.

A vector space $\mathcal{V}$ has two components: a set V of **vectors** and a field F of **scalars**, which will be represented by Greek letters.

In addition to the functions $+$ and $\times$ from $F \times F$ to F, there are also two functions: $\oplus$ from $V \times V$ to V, called vector addition, and $\otimes$ from $F \times V$ to V called scalar multiplication.

In its full generality a vector space should therefore be expressed as

$$\mathcal{V} = \{(V; \oplus), (F; +, \times); \otimes\} \ .$$

However it is usual to confuse $\mathcal{V}$ with V, and refer to V as the vector space.

Suppose now that the sets V and F have topologies $\mathcal{T}_V$ and $\mathcal{T}_F$ respectively.

For example, if $V = \mathbb{R}^n$ we have the topology induced by the Euclidean metric, and if $F = \mathbb{R}$ or $\mathbb{C}$ we have the usual topology.

If the functions $\oplus$, $\otimes$, $+$ and $\times$ are continuous with respect to these topologies and the appropriate induced product topologies we say that resulting entity is a **topological linear space.**

In practice, consideration is usually restricted to the real and complex fields, and to topologies on V induced by a **norm**.

A **norm** on a vector space is a real-valued function on V, whose value at $x \in V$ is denoted $||x||$, with the properties:

(a) $||x_1 \oplus x_2|| \leq ||x_1|| + ||x_2||$

(b) $||\alpha \otimes x|| = |\alpha| \times ||x||$

(c) $||x|| \geq 0$

(d) $||x|| \neq 0$ if $x \neq 0$

A vector space on which a norm is defined becomes a metric space, referred to as a **normed linear space**, if we define

$$d(x_1, x_2) = ||x_1 \ominus x_2|| \ .$$

Most of the examples of metrics considered earlier in the course fall into this category.

A function f from one normed linear space V to another normed linear space W is continuous at $x_0 \in V$ if, given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$||f(x) \ominus f(x_0)||_W < \epsilon \ \forall \ x \in V \ ; \ ||x \ominus x_0||_V < \delta \ .$$

The space is called *real* or *complex* depending on whether the field F is $\mathbb{R}$ or $\mathbb{C}$.

If U is a vector subspace of V, then the norm on V is also a norm on U, so that U is itself a normed linear space. This space is referred to as a *subspace* or *linear manifold*.

This subspace U may or may not be closed in V, and the distinction between these cases is often important.

Note that a linear space can be a metric space without being normed, and can be a topological space without being a metric space.

The inequalities

$$||(x_1 \oplus x_2) \ominus (y_1 \oplus y_2)|| \leq ||x_1 \ominus y_1|| + ||x_2 \ominus y_2||$$
$$||\alpha \otimes x \ominus \beta \otimes y|| \leq |\alpha| \times ||x \ominus y|| + |\alpha - \beta| \times ||y||$$

show that the operations $\oplus$ and $\otimes$ are continuous on a normed linear space.

As a consequence, if M is a linear manifold in X, so is its closure $Cl(M)$.

Note also, that since

$$|\,||x|| - ||y||\,| \leq ||x \ominus y||$$

the norm $||.||$ is a continuous function on V.

Notation. So far a notational distinction has been made between the operations of addition in F and V, and between scalar multiplication and multiplication in F.

However, common usage is to use the same symbol (+) for both forms of addition and no symbol at all for the two forms of multiplication. This usage will be used from here on.

When we are working with normed linear spaces, we are initially interested in functions which preserve the linear structure.

A function T from one normed linear space X to another normed linear space Y is called a **linear transformation** or **linear operator** if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)\ .$$

Note that this means that $T(0) = 0$.

If T is invertible, and both T and T^{-1} are continuous, then T induces a homeomorphism between X and Y, and the spaces are said to be *linearly homeomorphic* or *topologically isomorphic*.

If in addition, $||T(x)|| = ||x||$ for all $x \in X$, the spaces are said to be *isometrically isomorphic* or *congruent*.

Continuity of linear operators is an all-or-nothing affair.

Let T be a linear operator from X to Y.

Then T is continuous either at every point of X or at no point of X.

It is continuous on X if and only if there is a constant M such that $||T(x)|| \leq M||x||$ for every $x \in X$.

Let $x_0, x_1 \in X$, and suppose that T is continuous at x_0.

Then given any $\epsilon > 0$, there is $\delta > 0$ such that

$$||T(x) - T(x_0)|| < \epsilon\ \forall\ ||x - x_0|| < \delta\ .$$

Then

$$||T(y) - T(x_1)|| = ||T(y - x_1)||$$
$$= ||T((y - x_1 + x_0) - x_0)|| = ||T(y - x_1 + x_0) - T(x_0)|| < \epsilon$$
$$\forall\ ||(y - x_1 + x_0) - x_0|| < \delta\ ;\ \text{that is}\ \forall ||y - x_1|| < \delta$$

Therefore T is continuous at x_1 also, and hence throughout X.

(In fact T is uniformly continuous on X.)

If $||T(x)|| \leq M||x||$, then

$$||T(x)|| = ||T(x) - T(0)|| < \epsilon \ \forall \ ||x - 0|| = ||x|| < \frac{1}{M}\epsilon \ ,$$

so that T is continuous at $x_0 = 0$, and hence for all $x \in X$.

Conversely, if T is continuous on X, it is continuous at 0, and given $\epsilon = 1$, there is $\delta > 0$ such that

$$||T(x)|| = ||T(x) - T(0)|| < 1 \ \forall ||x|| < \delta \ .$$

For any $x \neq 0 \in X$, let

$$y = \frac{\delta}{2||x||}x$$

$||y|| = \frac{\delta}{2} < \delta$, so that

$$||T(y)|| = \left|\left|T\left(\frac{\delta}{2||x||}x\right)\right|\right| < 1$$

$$||T(x)|| = \frac{2||x||}{\delta}||T(y)|| < \frac{2}{\delta}||x|| \leq M||x||$$

where $M = \frac{2}{\delta}$.

Therefore we have $||T(x)|| \leq M||x||$ for $x \neq 0$, and the inequality is obviously true when $x = 0$ also.

As an example of a linear transformation which is not continuous, consider the set V of functions continuous on $[0, 1]$.

We can make this set into a normed linear space in a variety of ways.

Specifically, let us construct V_1 where the norm is

$$||f(t)|| = (||f||_1 =) \int_0^1 |f(t)| \, dt \ ,$$

and V_∞ where the norm is

$$||f(t)|| = (||f||_\infty =) \max_{0 \leq t \leq 1} |f(t)| \ .$$

The linear transformation given by the identity maping on V is not continuous from V_1 to V_∞.

Consider the functions

$$f_n(t) = \begin{cases} nt & 0 \leq t \leq \frac{1}{n} \\ 2 - nt & \frac{1}{n} < t < \frac{2}{n} \\ 0 & \frac{2}{n} \leq t \leq 1 \end{cases} \text{ for } n \geq 2 \ .$$

$$||f_n||_1 = \frac{1}{n} \ ; \ ||f_n||_\infty = 1$$

Therefore, given any $M > 0$, we can find $n > M$ such that

$$||T(f_n)|| = ||f_n||_\infty = 1 > \frac{M}{n} = M||f_n||_1 \ .$$

If we consider the vectors in X for which $||x|| \leq 1$ this inequality gives

$$||T(x)|| \leq M \ \forall \ ||x|| \leq 1 \ .$$

Therefore M is an upper bound for $\{||T(x)||, ||x|| \leq 1\}$.
Conversely, if M^* is an upper bound for these values, then for any $x \neq 0$

$$\left|\left|T\left(\frac{x}{||x||}\right)\right|\right| \leq M^*$$
$$||T(x)|| \leq M^*||x||$$

The least upper bound of these numbers is called the **norm** of the operator T, and

$$||T|| = \sup_{||x||\leq 1} ||T(x)|| = \sup_{||x||=1} ||T(x)|| = \sup_{x\neq 0} \frac{||T(x)||}{||x||} \ .$$

For example, consider the linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^m$, represented by the set of $m \times n$ matrices with complex coefficients.

Firstly, consider the norm on $\mathbb{C}^n$ given by

$$||x||_1 = \sum_{i=1}^{n} |\xi_i|$$

and the corresponding norm on $\mathbb{C}^m$.

$$\begin{aligned}
||Ax||_1 &= \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij}\xi_j \right| \\
&\leq \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|\,|\xi_j| \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{ij}| \right) |\xi_j| \\
&\leq \left(\max_j \sum_{i=1}^{m} |a_{ij}| \right) \sum_{j=1}^{n} |\xi_j| = \left(\max_j \sum_{i=1}^{m} |a_{ij}| \right) ||x||_1
\end{aligned}$$

and if this maximum occurs for column J, we have

$$||Ae_J||_1 = \sum_{j=1}^{n} |a_{iJ}| \ ,$$

so that this value is attained, and

$$||A||_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$$

Secondly, consider the norm

$$||x||_\infty = \max |\xi_i| \ .$$

$$\begin{aligned} ||Ax||_\infty &= \max_i \left| \sum_{j=1}^{n} a_{ij}\xi_j \right| \\ &\le \max_i \left(\sum_{j=1}^{n} |a_{ij}| \, |\xi_j| \right) \\ &\le \max_i \left(\sum_{j=1}^{n} |a_{ij}| \right) \max_j |\xi_j| = \max_i \left(\sum_{j=1}^{n} |a_{ij}| \right) ||x||_\infty \end{aligned}$$

and if this maximum occurs for row I, then choosing

$$x_j = e^{-i\theta} \ \text{ if } \ a_{Ij} = re^{i\theta}$$

shows that it is attained, and

$$||A||_\infty = \max_i \left(\sum_{j=1}^{n} |a_{ij}| \right)$$

Finally, if we use the Euclidean norm,

$$||Ax||_2^2 = x^* A^* Ax$$

where $*$ denotes the conjugate transpose.

The matrix A^*A is positive semidefinite Hermitian, so that there is a unitary transformation $x = Uy$, which is an isometry ($||y||_2 = ||x||_2$) and which reduces x^*A^*Ax to

$$\sum_{i=1}^{n} \lambda_i \eta_i^2$$

where the eigenvalues λ_i satisfy

$$\lambda_1 \ge \lambda_2 \ge \ldots \lambda_n \ge 0$$

Setting $\eta_1 = 1$ and the remaining values to 0, we see that

$$\lambda_1 = ||A||_2^2 \text{ and } ||A||_2 = \sqrt{\lambda_1}.$$

e.g. Consider

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} .$$

$$||A||_1 = \max(2,4) = 4$$

$$||A||_\infty = \max(1,5) = 5$$

$$\begin{aligned} A^*A &= \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 10 \end{pmatrix} \\ |tI - A^*A| &= \begin{vmatrix} t-4 & -6 \\ -6 & t-10 \end{vmatrix} = t^2 - 14t + 4 \\ \lambda_1 &= 7 + \sqrt{45} = 7 + 3\sqrt{5} \\ ||A||_2 &= \sqrt{\lambda_1} = \frac{3+\sqrt{5}}{\sqrt{2}} \simeq 3.7 \end{aligned}$$

If T is a continuous linear operator from X to Y, then if $S \subset \{||x|| \leq r\}$, $T(S) \subset \{||y|| \leq ||T||r\}$.

Hence $T(S)$ is bounded if S is bounded.

Conversely, if T is a linear operators with the property that $T(S)$ is bounded whenever S is bounded, then, in particular,

$$||T(x)|| \leq M \;\forall\; ||x|| \leq 1$$

and T is continuous.

There is a similar condition which determines invertibility.

Let T be a linear operator from X to Y.

The inverse T^{-1} exists and is continuous if and only if there is a constant $m > 0$ such that

$$m||x|| \leq ||T(x)||$$

for every $x \in X$.

If $x \neq 0$, $||x|| > 0$, so that $||T(x)|| > 0$, and $T(x) \neq 0$.

Therefore, if $T(x_1) = T(x_2)$, $T(x_1 - x_2) = 0$, and $x_1 = x_2$.

Hence T is $1-1$ and invertible.

If $y = T(x)$, $x = T^{-1}(y)$ and

$$\begin{aligned} m||T^{-1}(y)|| &\leq ||y|| \\ ||T^{-1}(y)|| &\leq \frac{1}{m}||y|| \end{aligned}$$

and T^{-1} is continuous.

Conversely, if T^{-1} exists and is continuous,

$$||T^{-1}(y)|| \leq M||y||$$
$$\frac{1}{M}||x|| \leq ||T(x)|| \ .$$

These two results together show that the normed linear spaces X and Y are topologically isomorphic (linearly homeomorphic) if and only if there is a linear operator T with domain X and range Y, and positive constants m and M such that

$$m||x|| \leq ||T(x)|| \leq M||x||$$

for every $x \in X$.

If we consider X and Y to have the same underlying vector space but different norms $||.||_X$ and $||.||_Y$, then choosing $T = I$ we see that these spaces are topologically equivalent if and only if there exist positive constants m and M such that

$$m||x||_X \leq ||x||_Y \leq M||x||_X$$

for every vector in the vector space.

Examples of Normed Linear Spaces.

We have already seen that for $p \geq 1$

$$||x||_p = (|\xi_1|^p + \cdots + |\xi_n|^p)^{1/p}$$

is a norm on $\mathbb{R}^n$.

The normed linear space obtained by using this norm is denoted $\ell^p(n)$.

When $p = 2$, we have the usual Euclidean distance function. In this case we have the alternative notation $\mathcal{E}^n$ for this space, and it is usual to assume that $\mathbb{R}^n$ uses this metric unless otherwise specified.

The limiting case as $p \to \infty$,

$$||x||_\infty = \max_{1 \leq i \leq n} |\xi_i|$$

defines the space $\ell^\infty(n)$.

These norms are extended to infinite sequences $\{\xi_i\}$.

The space ℓ^p consists of all such sequences for which

$$\sum_{i=1}^{\infty} |\xi_i|^p$$

converges. (Since these are sums of non-negative terms, this is equivalent to saying that this sum is bounded.)

The norm in ℓ^p is defined by

$$||x||_p = \left(\sum_{i=1}^{\infty} |\xi_i|^p \right)^{1/p} \ .$$

Similarly we have the space ℓ^∞ of bounded sequences for which

$$||x||_\infty = \sup_i |\xi_i| \ .$$

For any $x \in \ell^\infty$ we have

$$\sup_n |\xi_n| = \lim_{n\to\infty} \left[\lim_{p\to\infty} (|\xi_1|^p + \cdots + |\xi_n|^p)^{1/p} \right] \ .$$

If

$$\sum_{i=1}^\infty |\xi_i|^p$$

converges, then for some N, $|\xi_i| < 1$ for all $i > N$.

Therefore, if $q > p$, $|\xi_i|^q \le |\xi_i|^p$ for all $i > N$, and by the comparison test

$$\sum_{i=1}^\infty |\xi_i|^q$$

also converges.

Therefore $\ell^p \subset \ell^q$.

Consideration of the sequence

$$\left\{ \frac{1}{n^{1/p}} \right\}$$

which is in ℓ^q but not in ℓ^p shows that this is a proper inclusion.

It can be shown that if $x \in \ell^p$, $||x||_q \le ||x||_p$.

The space ℓ^p is complete.

Let $\{x_n\}$ be a Cauchy sequence in ℓ^p, with $x_n = \{\xi_i^{(n)}\}$.

For each k we have

$$\left|\xi_k^{(n)} - \xi_k^{(m)}\right| \le \left(\sum_{i=1}^\infty \left|\xi_i^{(n)} - \xi_i^{(m)}\right|^p \right)^{1/p} = ||x_n - x_m||_p$$

so that $\{\xi_k^{(n)}\}$ is a Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$, which converges to some element ξ_k.

Since $\{x_n\}$ is a Cauchy sequence, it is bounded; there is some constant B such that $||x_n|| \le B$ for all n.

For any finite M,

$$\left(\sum_{i=1}^M |\xi_i^{(n)}|^p \right)^{1/p} \le ||x_n|| \le B \ .$$

Letting $n \to \infty$ in this finite expression gives

$$\left(\sum_{i=1}^M |\xi_i|^p \right)^{1/p} \le B \ .$$

Since this is true for arbitrary M,

$$\left(\sum_{i=1}^\infty |\xi_i|^p \right)^{1/p} \le B \ ,$$

and $x = \{\xi_i\} \in \ell^p$.

It remains to show that $\{x_n\}$ converges to x.
Given any $\epsilon > 0$, we can find N such that for $m > n > N$, $||x_m - x_n|| < \frac{1}{2}\epsilon$.
Therefore, for any finite M,

$$\left(\sum_{i=1}^{M} |\xi_i^{(m)} - \xi_i^{(n)}|^p\right)^{1/p} \leq ||x_m - x_n|| < \frac{1}{2}\epsilon \ .$$

Again letting $m \to \infty$ in this finite expression, we obtain

$$\left(\sum_{i=1}^{M} |\xi_i - \xi_i^{(n)}|^p\right)^{1/p} \leq \frac{1}{2}\epsilon \ .$$

Since this is true for arbitrary M, it follows that

$$\left(\sum_{i=1}^{\infty} |\xi_i - \xi_i^{(n)}|^p\right)^{1/p} = ||x - x_n|| \leq \frac{1}{2}\epsilon < \epsilon$$

for all $n > N$, and the sequence converges to x.

Note This form of proof is common when proving completeness in linear spaces.

ℓ^p is an example of a **Banach Space**.
A Banach space is a complete normed linear space.

We have seen that if the metric space X is not complete, we can construct its completion $\hat{X}$.

If X and Y are normed linear spaces, and T is a continuous linear operator from X to Y, then there is a uniquely determined linear operator $\hat{T}$ from $\hat{X}$ to $\hat{Y}$ such that $\hat{T}(x) = T(x)$ if $x \in X$. Furthermore, $||\hat{T}|| = ||T||$.

For $\hat{x} \in \hat{X}$, consider a Cauchy sequence $\{x_n\}$ in X which defines $\hat{x}$.
Since

$$||T(x_n) - T(x_m)|| = ||T(x_n - x_m)|| \leq ||T|| \, ||x_n - x_m||$$

$\{T(x_n)\}$ is a Cauchy sequence in Y which defines an element $\hat{y} \in \hat{Y}$.
We set $\hat{T}(\hat{x}) = \hat{y}$.

If $x \in X$, the sequence $\{x\}$ shows that $\hat{T}(x) = T(x)$.

If $\{x_n\} \to \hat{x}$ and $\{y_n\} \to \hat{y}$, then $\{\alpha x_n + \beta y_n\} \to \alpha\hat{x} + \beta\hat{y}$.
Therefore

$$\hat{T}(\alpha\hat{x} + \beta\hat{y}) = \lim T(\alpha x_n + \beta y_n) = \lim(\alpha T(x_n) + \beta T(y_n)) = \alpha\hat{T}(\hat{x}) + \beta\hat{T}(\hat{y})$$

and $\hat{T}$ is linear.

If $\{x_n\} \to \hat{x}$, $||x_n|| \to ||\hat{x}||$. Furthermore

$$||T(x_n)|| \leq ||T|| \, ||x_n|| \text{ so that } ||\hat{T}(\hat{x})|| \leq ||T|| \, ||\hat{x}||$$

so that $\hat{T}$ is continuous, and $||\hat{T}|| \leq ||T||$.
Since for any $x \in X$,

$$||T(x)|| = ||\hat{T}(x)|| \leq ||\hat{T}|| \, ||x|| \ , ||T|| \leq ||\hat{T}|| \ .$$

Hence $||T|| = ||\hat{T}||$.

The spaces L^p.

Let $p \geq 1$.

The set $\mathcal{L}^p(a,b)$ is the class of functions x of a real variable s such that $x(s)$ is defined for all s, with the possible exception of a set of measure zero ('$x(s)$ is defined almost everywhere' (a.e.)) and is measurable and $|x(s)|^p$ is integrable in the sense of Lebesgue over the range (a,b). In this definition we can take $a = -\infty$ and/or $b = \infty$.

If $\mathcal{D}_x$ is the set on which x is defined, we define αx for $\alpha \in \mathbb{R}$ as the function such that

$$(\alpha x)(s) = \alpha(x(s)) \ \forall \ s \in \mathcal{D}_x$$

and $x + y$ as the function such that

$$(x+y)(s) = x(s) + y(s) \ \forall s \in \mathcal{D}_x \cap \mathcal{D}_y \ .$$

Clearly $\alpha x \in \mathcal{L}^p$ if $x \in \mathcal{L}^p$.

$x + y$ is obviously measurable. To show that $x + y \in \mathcal{L}^p$ if $x, y \in \mathcal{L}^p$, we need to show that $|x + y|^p$ is integrable.

Observe firstly that

$$\max\{|a|, |b|\} \leq |a| + |b| \leq 2\max\{|a|, |b|\}$$

Therefore

$$\begin{aligned} |a+b|^p &\leq (|a|+|b|)^p \leq (2\max\{|a|,|b|\})^p \\ &= \max\{2^p|a|^p, 2^p|b|^p\} \leq 2^p|a|^p + 2^p|b|^p \end{aligned}$$

Hence

$$|(x+y)(s)|^p \leq 2^p|x(s)|^p + 2^p|y(s)|^p$$

and since the left hand side is measurable and the right hand side is integrable, the left hand side is also integrable.

Unfortunately,

$$\int_a^b |x(s)|^p \, ds = 0$$

merely tells us that $x = 0$ almost everywhere on (a,b), so that

$$\left(\int_a^b |x(s)|^p \, ds \right)^{1/p}$$

is not a norm on $\mathcal{L}^p$.

We overcome this difficulty by defining the equivalence relation $=^0$ in $\mathcal{L}^p$ by $x =^0 y$ if $x = y$ almost everywhere.

The set of equivalence classes into which $\mathcal{L}^p$ is divided by this relation is denoted L^p.

If $[x]$ is the class in L^p to which $x \in \mathcal{L}^p$ belongs, we define the operations

$$[x] + [y] = [x + y]$$
$$\alpha[x] = [\alpha x]$$
$$||[x]||_p = \left(\int_a^b |x(s)|^p \, ds \right)^{1/p} \quad \text{for any } x \in [x]$$

on L^p.

With this structure, L^p is a normed linear space.

While the elements of L^p are in fact equivalence classes of functions, it is usual to write x instead of $[x]$

The case L^∞ requires separate treatment.

If (a, b) is a finite or infinite interval in $\mathbb{R}$, we say that a function x, measurable on (a, b), is *essentially bounded* if there is some $A \geq 0$ such that the set $\{t; |x(t)| > A\}$ has measure 0.

The set of all such constants is bounded below by 0, so that there will be a greatest lower bound for the set.

This smallest possible A is called the *essential least upper bound* of x, denoted $\sup^0 |x(t)|$.

If $\mathcal{L}^\infty$ is the class of all measurable and essentially bounded functions on (a, b), we construct L^∞ as before.

With the norm

$$||[x]|| = \sup^0 |x(t)| \text{for some } x \in [x]$$

L^∞ becomes a normed linear space.

Convergence in L_p.

When we consider a sequence $\{x_n\}$ of functions defined on some set S, there are a variety of modes of convergence which we can consider.

Firstly, we can consider the sequences $\{x_n(a)\}$ for each element $a \in S$. If the sequence converges for each such sequence, we say that the sequence converges pointwise on S.

Suppose that the pointwise limit defines a function x. If, given $\epsilon > 0$ we can find an integer N which depends on ϵ and S but not on the individual point a such that $|x_n(a) - x(a)| < \epsilon$ for $n > N$ for every $a \in S$, we say that the convergence is uniform. Convergence with respect to the sup norm is uniform.

The spaces L_p offer a different possibility. A sequence $\{x_n\}$ in L_p **converges in the mean (of order** p**)** to $x \in L_p$, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$||x_n - x||_p = \left(\int |x_n - x|^p \, ds \right)^{1/p} < \epsilon \ .$$

It is possible for a sequence of functions to converge in the mean but not pointwise.

To illustrate this last point, consider $L_p(0,1)$ and the intervals

$$[0,1],\ [0,\frac{1}{2}],\ [\frac{1}{2},1],\ [0,\frac{1}{3}],\ [\frac{1}{3},\frac{2}{3}]\ [\frac{2}{3},1],$$
$$[0,\frac{1}{4}],\ [\frac{1}{4},\frac{1}{2}],\ [\frac{1}{2},\frac{3}{4}],\ [\frac{3}{4},1],\ldots\ .$$

For the n^{th} interval in this sequence define

$$x_n = \begin{cases} 1 \text{ in the interval} \\ 0 \text{ otherwise} \end{cases} .$$

If $n \geq m(m+1)/2$ then $||x_n|| \leq (\frac{1}{m})^{1/p}$.
Therefore the sequence $\{x_n\}$ converges in L_p to $f = 0$.
However, for every $s \in S$, $f_n(s) = 1$ infinitely often and 0 infinitely often. Therefore the sequence does not converge pointwise.

It is also possible for a sequence of functions to converge uniformly but not in the mean. However, if the interval is finite, uniform convergence does imply convergence in the mean.

Linear functionals on L_p.
We begin with Hölder's inequality in its integral form;

$$\int |fg|\, ds \leq ||f||_p ||g||_q$$

where $\frac{1}{p} + \frac{1}{q} = 1$.
This states that if $f \in L_p$ and $g \in L_q$, $fg \in L_1$.
Suppose that we fix g in L_q.
For every f in L_p, $\int fg\, ds$ is defined, and

$$\int (\alpha f_1 + \beta f_2) g\, ds = \alpha \int f_1 g\, ds + \beta \int f_2 g\, ds$$

so that this defines a linear transformation from L_p into $\mathbb{R}$ or $\mathbb{C}$.
Linear transformations of this type are called linear functionals.
Furthermore

$$||T(f)|| = \left| \int fg\, ds \right| \leq \int |fg|\, ds \leq ||g||_q\, ||f||_p$$

so that this functional is continuous.

We could equally consider f fixed in L_p and let g vary in L_q giving a continuous linear functional on L_q.

On the other hand, if T is a continuous linear functional from L_p to $\mathbb{R}$, it can be shown that there is a $g \in L_q$ such that

$$T(f) = \int fg\, ds\ .$$

Of special interest is the case $p = q = 2$.
For $f, g \in L_2$, the linear functional

$$< f, g > = \int \bar{f} g\, ds$$

is called an inner product.

The spaces H^p.

Let $\mathfrak{U}$ denote the class of all functions $f(z)$ of a complex variable z which are analytic in the unit circle $|z| < 1$ (at least).

This class is a complex vector space.

We shall consider two examples of subspaces of $\mathfrak{U}$ which are normed linear spaces.

Firstly, for $p > 0$, $f \in \mathfrak{U}$ and $0 \leq r < 1$ we define

$$\mathfrak{M}_p[f;r] = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}$$

The class H^p consists of those $f \in \mathfrak{U}$ such that

$$\sup_{0 \leq r < 1} \mathfrak{M}_p[f;r] < \infty$$

It can be shown that for $p > 0$ these classes are closed under addition and scalar multiplication. Therefore they are subspaces of $\mathfrak{U}$.

If we define

$$||f|| = \sup_{0 \leq r < 1} \mathfrak{M}_p[f;r]$$

then this is a norm for $p \geq 1$, but not for $0 < p < 1$.

Thus H^p is a normed linear space for $p \geq 1$.

We extend our classes to H^∞, the class of bounded functions in $\mathfrak{U}$. The norm on H^∞ is

$$||f|| = \sup_{0 \leq r < 1} |f(z)|$$

For $1 \leq p < \infty$, if $f \in H^p$, then $\lim_{r \to 1-} f(re^{i\theta})$ exists for almost all values of θ, defining a function $f(e^{i\theta})$ which belongs to $L^p(0, 2\pi)$. Furthermore

$$||f|| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right)^{1/p}$$

which is the same as the norm in L^p.

Thus H^p is in isometric correspondence with a subset of L^p.

The other class of functions of interest is a subset of H^∞.

It consists of functions which are analytic in $|z| < 1$ and continuous on $|z| \leq 1$.

By the maximum modulus theorem for analytic functions,

$$||f|| = \max_{|z|=1} |f(z)|$$

for this class.

Finite-Dimensional Normed Linear Spaces.

Let X be a normed linear spaces of finite dimension $n \geq 1$. Then X is topologically isomorphic to $\ell^1(n)$ with the same scalar field.

Suppose that $x_1, x_2, \ldots, x_n$ is a basis for X.

Any vector $x \in X$ can be represented uniquely as $x = \sum_{i=1}^{n} \xi_i x_i$, and the mapping

$$x \leftrightarrow (\xi_1, \ldots, \xi_n)$$

is an isomorphism between X and $\ell^1(n)$.

$$||x|| = \left|\left| \sum_{i=1}^{n} \xi_i x_i \right|\right| \leq \sum_{i=1}^{n} |\xi_1| \, ||x_i|| \leq M \sum_{i=1}^{n} |\xi_i|$$

where $M = \max ||x_i||$.

It remains to prove that there is a constant $m > 0$ such that

$$m \sum_{i=1}^{n} |\xi_i| \leq \left|\left| \sum_{i=1}^{n} \xi_i x_i \right|\right|$$

If $\sum_{i=1}^{n} |\xi_i| = 0$, this is trivially true for any $m > 0$.

Otherwise let $\sum_{i=1}^{n} |\xi_i| = c > 0$.

The set $\{\sum_{i=1}^{n} |\xi_i| = c\}$ is compact in $\ell^1(n)$, and the mapping

$$f(\xi_1, \ldots \xi_n) = ||\xi_1 x_1 + \ldots \xi_n x_n||$$

is a continuous map from this set into $\mathbb{R}$.

Therefore this map **attains** its minimum mc on the set.

If $m = 0$, then $\sum \xi_i' x_i = 0$ for some set $(\xi_1', \ldots \xi_n') \neq (0, \ldots, 0)$, which implies that the vectors x_i are linearly dependent.

Therefore $m > 0$, and

$$m \sum_{i=1}^{n} |\xi_i| = mc \leq \left|\left| \sum_{i=1}^{n} \xi_i x_i \right|\right|$$

as required.

This means that any two normed linear spaces of the same finite dimension with the same scalar field are topologically isomorphic.

Since $\ell^1(n)$ is complete, it also follows that any finite-dimensional normed linear space is complete, and that any finite dimensional subspace of a normed linear space is closed.

Finally, since every closed and bounded set in $\ell^1(n)$ is compact, the same is true for every closed and bounded set in a finite dimensional normed linear space.

The converse of this result is also true.

Before proving this we have the following lemma.

Riesz's Lemma.

Suppose that X is a normed linear space.

Let X_0 be a subspace of X which is closed and a proper subset of X.

Then for each $0 < \theta < 1$ there exists a vector $x_\theta \in X$ such that $||x_\theta|| = 1$ and

$$||x - x_\theta|| \geq \theta$$

for all $x \in X_0$.

Select any $x_1 \in X \setminus X_0$, and let

$$d = \inf_{x \in X_0} ||x - x_1|| \ .$$

Since X_0 is closed, there is some ϵ neighbourhood of x_1 which contains no points of X_0, and it follows that $d > 0$.

Also, since $0 < \theta < 1$, $\theta^{-1}d > d$, and since d is the infimum there is some vector $x_0 \in X_0$ such that $||x_0 - x_1|| \leq \theta^{-1}d$.

Let $x_\theta = (x_1 - x_0)/||x_1 - x_0||$. Then $||x_\theta|| = 1$.

Since X_0 is a subspace, $x_0 + ||x_1 - x_0||x \in X_0$ for every $x \in X_0$, and so

$$||x - x_\theta|| = \left|\left|x - \frac{x_1 - x_0}{||x_1 - x_0||}\right|\right|$$

$$= \frac{1}{||x_1 - x_0||}||(x_0 + ||x_1 - x_0||x) - x_1|| \geq \frac{d}{||x_1 - x_0||} \geq \theta$$

for every $x \in X_0$.

Therefore, if X_0 is a closed and proper subspace of X, there are points on the unit sphere in X whose distance from X_0 is as near 1 as we please.

However, there may not be points on the unit sphere at distance 1 from X_0 as the following example shows.

Let X be the subspace of $C(0,1)$ consisting of all continuous functions on $[0,1]$ such that $x(0) = 0$.

For X_0 we take the subspace of all $x \in X$ such that

$$\int_0^1 x(t)\,dt = 0 \ .$$

Suppose that there is $x_1 \in X$ such that $||x_1|| = 1$, and $||x_1 - x|| \geq 1$ for all $x \in X_0$.

For each $y \in X \setminus X_0$ let

$$c = \int_0^1 x_1(t)\,dt \Big/ \int_0^1 y(t)\,dt$$

Then $x_1 - cy \in X_0$, and so

$$1 \leq ||x_1 - (x_1 - cy)|| = |c|\,||y||$$

or

$$\left| \int_0^1 y(t)\, dt \right| \le \left| \int_0^1 x_1(t)\, dt \right| \, ||y|| \ .$$

Consider the sequence $y_n(t) = t^{1/n}$.

For each n, $||y_n|| = 1$, and $\int_0^1 y_n(t)\, dt = \frac{n}{n+1}$.

Therefore

$$\left| \int_0^1 x_1(t)\, dt \right| \ge \frac{n}{n+1} \ \forall \ n \in N$$

and therefore

$$\left| \int_0^1 x_1(t)\, dt \right| \ge 1$$

But $x_1(0) = 0$, x_1 is continuous, and $|x_1| \le 1$ on $[0, 1]$, so that

$$\left| \int_0^1 x_1(t)\, dt \right| < 1 \ ,$$

and we have a contradiction.

Therefore no such x_1 exists.

We are now in a position to prove the converse of the earlier result:

Let X be a normed linear space, and suppose that the surface S of the unit sphere in X is compact. Then X is finite dimensional.

Suppose that X is not finite dimensional.

Choose $x_1 \in S$, and let X_1 be the subspace (of dimension 1) generated by x_1. Since X is not of finite dimension, X_1 is a proper subspace of X, and because it is finite dimensional it is closed.

Hence, by Riesz's lemma, there exists $x_2 \in S$ such that $||x_2 - x_1|| \ge \frac{1}{2}$.

Let X_2 be the (closed and proper) subspace of X generated by x_1, x_2; then there exists $x_3 \in S$ such that $||x_3 - x|| \ge \frac{1}{2}$ for all $x \in X_2$. In particular $||x_3 - x_1|| \ge \frac{1}{2}$ and $||x_3 - x_2|| \ge \frac{1}{2}$.

Proceding by induction, we obtain an infinite sequence of elements of S such that $||x_n - x_m|| \ge \frac{1}{2}$ if $m \ne n$.

This sequence can have no convergent subsequence. This contradicts the requirement that S be compact. Thus X must be finite dimensional.

Alternatively, we could proceed as follows.

Consider the set of open spheres with radius $\frac{1}{2}$ and centres in S. This is an open cover for S, and hence there is a finite set $x_1, \ldots, x_n$ of points on S such that every point $x \in S$ is distance less than $\frac{1}{2}$ from some x_i.

Let M be the space generated by the points x_i. It is finite dimensional and therefore closed.

If M is a proper subset of X, then by Riesz's lemma there is a point $x \in S$ whose distance from every point of M is greater than or equal to $\frac{1}{2}$.

Since this is impossible, $M = X$, and X is finite dimensional.