MATH 3402

METRIC SPACE TOPOLOGY

Open sets.

A subset S of the set X is **open** in the metric space (X, d), if for every $x \in S$ there is an $\epsilon_x > 0$ such that the ϵ_x neighbourhood of x is contained in S.

That is, for every $x \in S$; if $y \in X$ and $d(y, x) < \epsilon_x$, then $y \in S$.

An ϵ neighbourhood is open. It is often referred to as an "open ϵ-neighbourhood" or "open ϵ-ball".

For $a \in X$, consider the set $\mathcal{N}(a, \epsilon) = \{x \in X; d(x, a) < \epsilon\}$.

If $x \in \mathcal{N}$, then $d(x, a) = r < \epsilon$.

Therefore $\epsilon_x = \epsilon - r > 0$, and for all $y \in X$ with $d(y, x) < \epsilon_x$,

$$d(y, a) \leq d(y, x) + d(x, a) < \epsilon_x + r = \epsilon \ ,$$

so that $y \in \mathcal{N}$.

X is open in (X, d).

ϕ is open in (X, d).

If the metric d is the discrete metric, then every subset S of X is open in (X, d).

For any $x \in S$ the neighbourhood $\mathcal{N}(x, \frac{1}{2}) = \{x\}$, which is trivially in S.

The union of any number of open sets in (X, d) is open in (X, d).

$$\text{If} \quad x \in \bigcup_{\alpha} S_\alpha$$

then $x \in S_a$ for some $\alpha = a$.

Since S_a is open, there is $\mathcal{N}(x, \epsilon) \subset S_a$.

$$\text{Therefore} \quad \mathcal{N}(x, \epsilon) \subset \bigcup_{\alpha} S_\alpha \ .$$

The intersection of a finite number of open sets in (X, d) is open in (X, d).

$$\text{If} \quad x \in \bigcap_{i=1}^{n} S_i$$

then $x \in S_i$ for each i.

Since S_i is open, there is an $\epsilon_i > 0$ such that $\mathcal{N}(x, \epsilon_i) \subset S_i$.

Let $\epsilon = \min_i \epsilon_i$. Then $\epsilon > 0$, and

$$\mathcal{N}(x, \epsilon) \subset \mathcal{N}(x, \epsilon_i) \subset S_i \text{ for each } \ i$$

Therefore

$$\mathcal{N}(x, \epsilon) \subset \bigcap_{i=1}^{n} S_i \ .$$

If there are infinitely many sets in the intersection, then the min is replaced by the inf which may be zero.

For example, the sets

$$S_n = \{|z - z_0| < R + \frac{1}{n}\}$$

are open in $(\mathbb{C}, |.|)$, but

$$\bigcap_{n=1}^{\infty} S_n = \{|z - z_0| \le R\}$$

is not.

Closed sets.

Definition: a is an accumulation point of the set S if every ϵ neighbourhood of a contains a point $(x \neq a) \in S$.

A subset S of X is **closed** in (X, d) if S contains all its accumulation points.

S is closed in (X, d) if and only if $\backslash S$ is open in (X, d).
The complement is taken with respect to the set X.

If $x \in \backslash S$, then x is not in S and x is not an accumulation point of S.
Therefore there is some $\epsilon > 0$ such that $\mathcal{N}(x, \epsilon)$ contains no points of S.
Therefore $\mathcal{N}(x, \epsilon) \subset \backslash S$, and $\backslash S$ is open.

Conversely, if $\backslash S$ is open, then if $x \in \backslash S$, there is an epsilon neighbourhood of x such that $\mathcal{N}(x, \epsilon) \subset \backslash S$.
Therefore x is not an accumulation point of S, and hence every accumulation point of S belongs to S.

From de Morgan's laws it follows that the intersection of any number of closed sets is closed, and the union of any finite number of closed sets is closed.

As with the case of open sets, the union of an infinite number of closed sets need not be closed.
The sets

$$S_n = \{|z - z_0| \le R - \frac{1}{n}\}$$

are closed in $(\mathbb{C}, |.|)$, but

$$\bigcup_{n=1}^{\infty} S_n = \{|z - z_0| < R$$

is not.

X is closed in (X, d).
ϕ is closed in (x, d).
If d is the discrete metric, every subset of (X, d) is closed.

Continuity.

A function f from (X, d_X) to (Y, d_Y) is continuous at $a \in X$, if, given any $\epsilon > 0$, there exists a $\delta(\epsilon, a)$ such that

$$d_Y(f(x), f(a)) < \epsilon \ \forall \ x \in X \ ; d_X(x, a) < \delta \ .$$

In terms of open sets, this says that

$$f(\mathcal{N}(a, \delta)) \subset \mathcal{N}(f(a), \epsilon) \ .$$

A function f from (X, d_X) to (Y, d_Y) is continuous at $a \in X$ if and only if for any sequence $\{a_n\}$ in (X, d_X) which converges to a, the sequence $\{f(a_n)\}$ converges in (Y, d_Y) to $f(a)$.

A function f from (X, d_X) to (Y, d_Y) is continuous on X if it is continuous at every point of X.

Theorem. The function f from (X, d_X) to (Y, d_Y) is continuous on X if and only if for every open set U in Y, $f^{-1}(U)$ is open in X.

Proof.

a) Suppose that f is continuous on X, and let U be open in Y.

If $a \in f^{-1}(U)$, then $f(a) \in U$.

Since U is open, there is some $\epsilon > 0$ such that $\mathcal{N}(f(a), \epsilon) \subset U$.

But f is continuous at a. Therefore there is a $\delta > 0$ such that $f(\mathcal{N}(a, \delta)) \subset \mathcal{N}(f(a), \epsilon) \subset U$.

Therefore $\mathcal{N}(a, \delta) \subset f^{-1}(U)$, and $f^{-1}(U)$ is open.

b) Conversely, suppose that $f^{-1}(U)$ is open in (X, d_X) for every U open in (Y, d_Y).

For any $a \in X$, $\mathcal{N}(f(a), \epsilon)$ is open in Y.

Therefore $f^{-1}(\mathcal{N}(f(a), \epsilon))$ is open in X, and a is in this set.

Therefore for some $\delta > 0$, $\mathcal{N}(a, \delta) \subset f^{-1}(\mathcal{N}(f(a), \epsilon))$ and hence $f(\mathcal{N}(a, \delta)) \subset \mathcal{N}(f(a), \epsilon)$.

This result shows that continuity of a function f from X to Y is determined by the open sets in X and Y.

If therefore two metrics on X give rise to precisely the same open sets in X, then any function continuous with respect to one metric will be continuous with respect to the other.

Definition. Two metrics d_1 and d_2 on a space X are **topologically equivalent** if and only if a subset U of X which is open in (X, d_1) is open in (X, d_2).

If d_1 and d_2 are topologically equivalent on X, then

a) $f : X \to Y$ is continuous from (X, d_1) to (Y, d_Y) if and only if it is continuous from (X, d_2) to (Y, d_Y).

b) $f : Y \to X$ is continuous from (Y, d_Y) to (X, d_1) if and only if it is continuous from (Y, d_Y) to (X, d_2).

Proof.

a): Suppose that d_1 and d_2 are topologically equivalent, and that f is continuous from (X, d_1) to (Y, d_Y).

Then , for every open set U in (Y, d_Y), $f^{-1}(U)$ is open in (X, d_1).

Since d_1 and d_2 are topologically equivalent, $f^{-1}(U)$ is also open in (X, d_2), so that f is continuous from (X, d_2) to (Y, d_Y).

Conversely, suppose that f continuous from (X, d_1) to (Y, d_Y) implies f continuous from (X, d_2) to (Y, d_Y).

Since this does not depend on f or Y, we are free to choose the image space as (X, d_1) and the function from X to X as the identity function.

Then for any open set U in (X, d_1), $f^{-1}(U) = U$ is open in both (X, d_1) and (X, d_2), and d_1 and d_2 are topologically equivalent.

b): Follows in similar fashion.

Theorem. If there are strictly positive constants c_1 and c_2 such that

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y)$$

for all $x, y \in X$, then d_1 and d_2 are topologically equivalent metrics on X.

Proof.

Let U be open in (X, d_1).

Then for $a \in X$ there exists $\epsilon > 0$ such that

$$\{x \in X; d_1(x,a) < \epsilon\} \subset U \ .$$

But then

$$\{x \in X; d_2(x,a) < c_1\epsilon\} \subset \{x \in X; d_1(x,a) < \epsilon\} \subset U$$

and U is open in (X, d_2).

Similarly, if U is open in (X, d_2) it is open in (X, d_1), and the metrics are topologically equivalent.

This criterion is sufficient but not necessary.

Let $X = \mathbb{R}$, and consider the metrics

$$\begin{aligned} d_1(x,y) &= |x-y| \\ d_2(x,y) &= \frac{|x-y|}{1+|x-y|} \end{aligned}$$

Obviously $d_2(x,y) \leq d_1(x,y)$ for all x, y, but since $d_1(x,y) = (1+|x-y|)d_2(x,y)$ there is no strictly positive constant c such that $d_2(x,y) \geq c d_1(x,y)$.

On the other hand

$$\{d_1(x,a) < \epsilon\} \subset \{d_2(x,a) < \epsilon\}$$

so that if U is open in $(\mathbb{R}, d_2)$ it is open in $(\mathbb{R}, d_1)$, while if $|x-a| < \epsilon$

$$\begin{aligned} d_1(x,a) &= (1+|x-a|)d_2(x,a) \\ &\leq (1+\epsilon)d_2(x,a) \end{aligned}$$

Let $\epsilon_1 = \epsilon/(1+\epsilon)$.

Then $\{d_2(x,a) < \epsilon_1\} \subset \{d_1(x,a) < \epsilon\}$, so that if U is open in $(\mathbb{R}, d_1)$ it is open in $(\mathbb{R}, d_2)$.

For example, in $\mathbb{R}^2$,

$$\begin{aligned} &\max(|x_1 - x_2|, |y_1 - y_2|) \\ \leq\ &|x_1 - x_2| + |y_1 - y_2| \\ \leq\ &2\max(|x_1 - x_2|, |y_1 - y_2|) \end{aligned}$$

so that the taxi-cab and sup metrics are equivalent.

In fact all the metrics generated by the norms $||x||_p$ are topologically equivalent on $\mathbb{R}^n$ for finite n.