ASSIGNMENT 2 FOR MATH3402 IN 2013

Due date: 12 April 2013.

PLEASE SUBMIT IT TO THE ASSIGNMENT BOX,
LEVEL FOUR, PRIESTLEY BUILDING #67

Question 1. *(3 marks)*

Let $S = C^0([0,1];\mathbb{R})$ *denote the set of all continuous functions* $f : [0,1] \to \mathbb{R}$. *For two continuous functions* $f \in S$, $g \in S$, *set*

$$d(f,g) = \max_{0 \le x \le 1} |f(x) - g(x)|.$$

Prove (S,d) *is a metric space.*

Question 2. *(3 marks)*

(i) Let (S,d) *be a metric space (i.e. d is a metric on* S*). Show that for any* x, y, $z \in S$

$$|d(x,z) - d(y,z)| \le d(x,y)$$

(ii) Let (S,d) *be a metric space. Take* $a \in S$ *and let* $f : S \to \mathbb{R}$ *be a function defined by*

$$f(x) = d(x,a), \quad \forall x \in S.$$

Then show that f *is continuous at each* $x_0 \in S$.

Question 3. *(4 marks)*

Let $p > 1$, $q > 1$ *be the dual indices, i.e.* $\frac{1}{p} + \frac{1}{q} = 1$ *and let* $X = C^0([a,b];\mathbb{R})$ *be the space of all continuous functions on* $[a,b]$ *with two real numbers* $a < b$. *Assume* $f(x)$ *and* $g(x)$ *be continuous functions on* $[a,b]$ *i.e.* $f, g \in C^0([a,b];\mathbb{R})$. *Then*

(i) Use Young's inequality to prove

$$\int_a^b |f(x)|\,|g(x)|\,dx \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} \left(\int_a^b |g(x)|^q\,dx\right)^{1/q} \tag{3.1}$$

where (3.1) is called "Hölder's inequality".

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX *(ii) Use the inequality (3.1) to prove*

$$\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/p} \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p} \tag{2.2}$$

(iii) Use the inequality (3.2) to prove that

$$d(f,g) = \|f-g\|_p = \left(\int_a^b |f(x)-g(x)|^p\,dx\right)^{1/p}$$

is a metric on the space $C^0([a,b];\mathbb{R})$ *of all continuous functions on* $[a,b]$.