MATH 3402

TUTORIAL SHEET 9

1. If

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

show that

$$||A||_2 = \frac{1}{2}\left(|b| + \sqrt{|b|^2 + 4|a|^2}\right) \ ,$$

Ans.

$$A^*A = \begin{pmatrix} a^* & 0 \\ b^* & a^* \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} |a|^2 & a^*b \\ b^*a & |b|^2 + |a|^2 \end{pmatrix}$$

$$|tI - A^*A| = (t - |a|^2)(t - (|a|^2 + |b|^2)) - |a|^2|b|^2 = t^2 - (2|a|^2 + |b|^2)t + |a|^4$$

$$t_+ = \frac{1}{2}(2|a|^2 + |b|^2 + \sqrt{|b|^4 + 4|a|^2|b|^2} = \frac{1}{4}\left(\sqrt{|b|^2 + 4|a|^2} + |b|\right)^2$$

$$||A||_2 = \frac{1}{2}\left(|b| + \sqrt{|b|^2 + 4|a|^2}\right)$$

and if

$$A = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$$

then

$$||A||_2 = \frac{1}{2}\left(\sqrt{(|a| + 1)^2 + |b|^2} + \sqrt{(|a| - 1)^2 + |b|^2}\right) \ .$$

Ans.

$$A^*A = \begin{pmatrix} 0 & a^* \\ 1 & b^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} = \begin{pmatrix} |a|^2 & a^*b \\ b^*a & 1 + |b|^2 \end{pmatrix}$$

$$|tI - A^*A| = (t - |a|^2)(t - (1 + |b|^2)) - |a|^2|b|^2 = t^2 - (1 + |a|^2 + |b|^2)t + |a|^2$$

$$t_+ = \frac{1}{2}\left(1 + |a|^2 + |b|^2 + \sqrt{(|a|^2 + 2|a| + 1 + |b|^2)(|a|^2 - 2|a| + 1 + |b|^2)}\right)$$

$$= \frac{1}{4}\left(\sqrt{(|a| + 1)^2 + |b|^2} + \sqrt{(|a| - 1)^2 + |b|^2}\right)^2$$

$$||A||_2 = \frac{1}{2}\left(\sqrt{(|a| + 1)^2 + |b|^2} + \sqrt{(|a| - 1)^2 + |b|^2}\right)$$

These two results are particular cases of

$$\left|\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|\right|_2 = \frac{1}{2}\left(\sqrt{S^2 + 2|\Delta|} + \sqrt{S^2 - 2|\Delta|}\right)$$

where

$$S^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2$$
$$\text{and } \Delta = ad - bc$$

In fact, we have

$$m||x|| \le ||Ax|| \le M||x||$$
$$\text{where } m == \frac{1}{2}\left(\sqrt{S^2 + 2|\Delta|} - \sqrt{S^2 - 2|\Delta|}\right)$$

and A is invertible iff $\Delta \neq 0$.

2. If $C(0,1)$ is the set of functions continuous on $[0,1]$ with the uniform metric, and $D(0,1)$ is the set of continuously differentiable functions on $[0,1]$ with the same metric;

(a) Is $T : C \to D$ given by $T(f)(x) = \int_0^x f(t)\,dt$ continuous?

Ans. If $||f|| \le 1$,

$$\left|\int_0^x f(t)\,dt\right| \le \int_0^x |f(t)|\,dt \le x$$
$$\left|\left|\int_0^x f(t)\,dt\right|\right| \le 1$$
$$||T|| = 1$$

and T is continuous.

(b) Is $T : D \to C$ given by $T(f)(x) = f'(x)$ continuous?

Ans. Consider $f_n = \sin nx \in D$

$||f_n|| = 1$, but $||f_n'|| = ||n \cos nx|| = n$.

Therefore, for any $M > 0$, if $n > M$

$$||T(f_n)|| > M||f_n||$$

and T is not continuous.

These results are a variant of the results that a uniformly convergent series can be integrated term by term but not necessarily differentiated term by term.

3. Let T be a linear transformation from ℓ^1 to ℓ^1.

Set $e_i = \{\delta_{ij}\}$ and $a_i = T(e_i)$.

Show that $||T|| = \sup_i ||a_i||_1$.

Ans.

For $x = \{\xi_1\} \in \ell^1$, let $x_n = \sum_{i=1}^n \xi_i e_i$.

$$T(x_n) = \sum_{i=1}^n \xi_i T(e_i)$$
$$||T(x_n)|| \le \sum_{i=1}^n |\xi_i|\,||a_i|| \le (\sup_i ||a_i||) \sum_{i=1}^n |\xi_i|$$

Taking the limit as $n \to \infty$, we have

$$||T(x_n)|| \le (\sup_i ||a_i||)||x||$$

For any $\epsilon > 0$, we can find a_I such that

$$||a_I|| > \sup_i ||a_i|| - \epsilon$$

so that

$$||T(e_I)|| > (\sup_i ||a_i|| - \epsilon)||e_I||$$

Therefore

$$(\sup_i ||a_i|| - \epsilon) < ||T|| \le \sup_i ||a_i||$$

and the result follows.

4. Let X be a finite dimensional normed linear space, and Y a normed linear space.

If T is a linear operator from X to Y, show that T is continuous.

Ans.
Since X is finite dimensional, X is linearly homeomorphic to $\ell^1(n)$.
Let f be the continuous invertible linear function defining the homeomorphism.
We can represent $T : X \to Y$ as $S = T \circ f : \ell^1 \to Y$, where S is a linear transformation.
If (e_i) is the basis for ℓ^1 and $a_i = S(e_i)$,

$$||S(x)|| = ||S(\{\xi_i\}|| \leq \sum_{I=1}^{n} |\xi_i|\, ||a_i|| \leq \left(\max_i ||a_i||\right) ||x||$$

Therefore S is continuous and so $T = S \circ f^{-1}$ is also continuous.