MATH 3402

TUTORIAL SHEET 1

1. Describe each of the following sets as the empty set, as $\mathbb{R}$, or in interval notation as appropriate:

(a) $$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)$$

(b) $$\bigcup_{n=1}^{\infty}(-n,n)$$

(c) $$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},1+\frac{1}{n}\right)$$

(d) $$\bigcup_{n=1}^{\infty}\left(-\frac{1}{n},2+\frac{1}{n}\right)$$

(e) $$\bigcup_{n=1}^{\infty}\left(\mathbb{R}\setminus\left(-\frac{1}{n},\frac{1}{n}\right)\right)$$

(f) $$\bigcap_{n=1}^{\infty}\left(\mathbb{R}\setminus\left[\frac{1}{n},2+\frac{1}{n}\right]\right)$$

2. Show that if $A \subset B \subset \mathbb{R}$, and if B is bounded above, then A is bounded above, and $\sup A \le \sup B$.

3. Let a_0 and a_1 be distinct real numbers.
Define $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for each positive integer $n \ge 2$.
Show that $\{a_n\}$ is a Cauchy sequence.

4. Suppose x is an accumulation point of $\{a_n : n \in \mathbb{N}\}$.
Show that there is a subsequence of $\{a_n\}$ that converges to x.

5. Given the non-negative real numbers $a_1, a_2, \ldots, a_r$, let $a = \sup\{a_i\}$.
Prove that for any integer n,

$$a^n \le a_1^n + a_2^n + \cdots + a_r^n \le ra^n \ ,$$

and determine

$$\lim_{n\to\infty}(a_1^n + a_2^n + \cdots + a_r^n)^{1/n} \ .$$

6. Consider the sequence

$$0,\ 1,\ -1,\ 2,\ -2,\ \frac{1}{2},\ -\frac{1}{2},\ \ldots$$

used to demonstrate the countability of the rationals.
a) What is the fiftieth term in this sequence?
b) Which term in the sequence is $-\frac{4}{5}$?