MATH 3402

Tutorial Sheet 2

1. In each of the following cases, say whether (X, d) is a metric space or not. If it is not, say which of the axioms fails.

(a)
$X = \mathbb{R}^2$; $d((x,y),(x',y')) = |y - y'|$.

(b)
$X = \mathbb{C}$; $d(z_1, z_2) = |z_1 - z_2|$.

(c)
$X = \mathbb{Q}$; $d(x,y) = (x-y)^3$.

(d)
$X = \mathbb{C}$; $d(z_1, z_2) = \min\{|z_1| + |z_2|, |z_1 - 1| + |z_2 - 1|\}$ if $z_1 \neq z_2$, $d(z,z) = 0$.

(e)

$$X = \mathbb{R} \; ; \; d(x,y) = \left| \int_x^y f(t)\, dt \right| ,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given positive integrable function.

2. Sketch the sets $\{\underline{x} \in \mathbb{R}^2, d(\underline{x}, \underline{0}) < 1\}$ when $d(\underline{x}, \underline{y})$ is
(a) The Euclidean metric;
(b) The taxicab metric;
(c) The sup metric;
(d) The discrete metric.

3. For $z_j = x_j + iy_j \in \mathbb{C}$, let

$$\xi_j = \frac{2x_j}{1 + |z_j|^2} \; ; \; \eta_j = \frac{2y_j}{1 + |z_j|^2} \; ; \; \zeta_j = \frac{1 - |z_j|^2}{1 + |z_j|^2} \; .$$

Show that

(i)
$$\xi_j^2 + \eta_j^2 + \zeta_j^2 = 1$$

(ii)
$$\left((\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2\right)^{1/2} = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2}\sqrt{(1 + |z_2|^2)}}$$

Hence deduce that

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2}\sqrt{(1 + |z_2|^2)}}$$

is a metric on $\mathbb{C}$.

Show that the sequence $\{z_n = n\}$ is a Cauchy sequence with respect to this metric.

4. Show that if for some $x_0 \in S$, $d(x, x_0) < k$ for all $x \in Q$, then for any $a \in S$ there is a constant k_a such that $d(x, a) < k_a$ for all $x \in Q$.
(That is, a set Q is bounded in S if it is bounded with respect to some member of S.)