MATH 3402

TUTORIAL SHEET 3

1. Show that a finite union of bounded sets is bounded.

Show that the intersection of an arbitrary number of bounded sets is either bounded or empty.

2. Show that if a metric space (X, d) has the property that every bounded sequence converges, then X consists of only one point.

3. Say whether the following sequences converge, and find their limit if they do:

a) $a_n = x^{-n}$ in $C(\frac{1}{3}, \frac{2}{3})$ (the continuous functions on the open interval $(\frac{1}{3}, \frac{2}{3})$) with the uniform - sup - metric;

b) $a_n = e^{-nx}$ in $C[0, 1]$ with the uniform metric.

c) $a_n = (\alpha_n, f(\alpha_n))$ in $\mathbb{R}$ with the Euclidean metric, where α_n is a convergent sequence in $\mathbb{R}$ with limit α, and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

4. Determine the diameter of the set $\{|z| < r\}$ in $(\mathbb{C}, d)$ where

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}$$

Repeat the exercise for the set $\{|z - 1| < r\}$.

5. Let $\{f_n(x)\}$ be a sequence of functions, continuous on the closed interval $[a, b] \in \mathbb{R}$, which is a Cauchy sequence with respect to the uniform metric on $C[a, b]$.

a) Prove that the sequence converges pointwise on $[a, b]$.

b) Prove that the function defined by these pointwise limits is continuous on $[a, b]$.

6. Show that the taxicab and Euclidean metrics are topologically equivalent on $\mathbb{R}^2$.

7. Show that the metrics

$$d_1(z_1, z_2) = |z_1 - z_2|$$

and

$$d_2 = \frac{2|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}$$

are topologically equivalent on $\mathbb{C}$.