

Chapter 2

Rational pricing

So far our financial world consists of a single type of stock with price process $S = (S_t, 0 \leq t \leq T)$, and a few financial contracts. As yet we have made no assumptions about the possible paths S may take. The results of this chapter hold independently of the model we choose. Gradually we will build up into examining some sophisticated models. The next refinement is to include the time value of money.

We will assume the existence of a *risk-free interest rate*. All market participants may invest and borrow at this same rate r . For now we assume r to be constant. In reality, of course, interest rates vary throughout time.

Everyone will remember how compound interest accumulates. If I have M dollars invested at a rate of r percent per annum compounded n times per year, then after one year my account will contain

$$M \left(1 + \frac{r}{n}\right)^n = M e^{n \log(1+r/n)}$$

dollars. The expansion for $n \log(1+r/n)$ is $r + o(1/n)$, where $o(h)$ represents a function of h such that $\frac{o(h)}{h} \rightarrow 0$ as $h \rightarrow 0$. Thus, in the limit as the number of compounding periods in a year n tends to infinity, after one year my original investment M will have grown to Me^r dollars. This is called *continuously compounding interest*, and is quite commonly used as it is simpler to incorporate variable interest rates into a continuously compounding interest model than a discrete model.

So M dollars invested at this continuously compounded rate of interest will grow

to be Me^{rT} in T years. Similarly, the value now of M dollars promised in T years time is Me^{-rT} . Taking r to be the interest rate for both borrowing and lending is not such a restrictive assumption as it seems. Usually the agent is either borrowing or lending (not both) and knows in advance which they will be doing.

Also in this chapter we will use the ability to sell short. Someone is said to be *selling short* if they sell stock that they don't actually have. Effectively short selling allows an investor to stack a portfolio with negative amounts of various assets. In reality only certain assets may be sold short and these usually have a time limit within which they must be either bought back or delivered. Our assumption will be that arbitrarily large quantities of any asset may be sold short with no limitations on the time to delivery.

2.1 Arbitrage

Suppose we are offered a forward contract on some stock. Recall a forward contract is the trade of stock at some future date T for a price K specified now (at time 0). *What is the future price we should agree to?*

My first guess was to model the stock price at expiry S_T as a random variable with an assumed distribution. And then use the expected value $\mathbb{E}(S_T)$ as my strike price. According to the theory of arbitrage, using the expected value method I would only be right by coincidence.

Consider the following strategy by the seller: borrow enough money to buy the stock now, wait until the date the contract matures, deliver the stock in exchange for the strike price as promised, and pay back the loan and interest accrued. Now if the strike price is greater than our borrowings we will have made a risk-free profit, and it is unlikely that the buyer would have agreed to this. Therefore we have

$$K \leq S_0 e^{rT}.$$

A similar tactic can be used by the buyer: sell the stock short and invest the cash at the risk-free interest rate. When the contract matures pay the strike price, receive the stock from the seller, and pass it on to cover the original sale. If the cash plus

interest earned is greater than the strike price, the buyer would make money with certainty. Assuming the seller isn't silly

$$K \geq S_0 e^{rT}.$$

Thus the strike price for the forward contract must be equal to the cost of the stock initially plus the interest on this amount over the duration of the contract $K = S_0 e^{rT}$. The buyer and seller cannot agree on any other price.

This is the classic example of an arbitrage argument. If the strike price K was quoted at any price other than $S_0 e^{rT}$ then there would be an opportunity for one side to make a riskless profit. Such a situation is called an arbitrage opportunity. An *arbitrage opportunity* is said to exist whenever there is the possibility for profit with no risk of loss. Note that profit does not need to occur with certainty, but there must be no possibility for loss. An arbitrage opportunity could be as simple as a price discrepancy between two traders of the same stock or currency; buying from one and selling to the other will result in a risk-free profit.

So why does the expectation approach fail to give the correct price? If K , set according to the arbitrage argument, turned out to be lower than $\mathbb{E}(S_T)$ then entering into the forward contract as a buyer would be a good move because, to the best of our model's predictions, we will make a profit of $\mathbb{E}(S_T) - K$. In fact, after playing this game many times, the strong law of large numbers¹ dictates that our average payoff per game will tend to $\mathbb{E}(S_T) - K$ almost surely (with probability 1). The problem is that the laws of large numbers don't apply when only a single play is allowed, as the case is here. Entering into a forward contract with an expected payoff of $\mathbb{E}(S_T) - K > 0$ is still a good move, but then again the investors which borrow S_0 to buy the actual stock also expect to make just as much profit.

Arbitrage enforces not only the strike price of a forward contract, but it also sets the price of other claims as well. Actually it wasn't until Black & Scholes (1973) famous paper that this fact became apparent. Before 1973, few people would have disagreed with a price quoted for a call option using expectation and the law of large numbers. Now, any price other than that determined by arbitrage is quickly taken advantage of as a source of free money.

¹references include Grimmett & Stirzaker (1997), Ross (1996), and Dudley (1989)

The usual assumption regarding arbitrage opportunities is that there are none. That is to say, the market model is *arbitrage-free*. This is not unreasonable since when price discrepancies do exist in the real world, the demand and supply become vastly different. Consequently the prices are driven back to where they should be.

Often the arbitrage-free assumption is strengthened to that of completeness. In a *complete market* the payoff for any claim can be replicated by a trading strategy involving stocks and bonds; the same way as we decided the correct strike price of a forward. In a complete market there is a unique price for every contract and this is equal to the initial cost of setting up a portfolio which will replicate the claim.

This result is of fundamental importance. It has been written in various different forms in the literature. Some of the particularly appealing presentations include Theorem 11.9 and lead up results of Klebaner (1998), Section 1.4.4 of Musiela & Rutkowski (1997), and for a more technical treatment see Sections 12.1 and 12.2 of Øksendal (1998).

Consistent with the idea of an arbitrage-free market is the notion of rationality among investors. That is to say, if all investors prefer more wealth to less then no-one will agree to be on the downside of an arbitrage opportunity.

Consider two portfolios A and B . These may contain any combination of stocks, bonds, or contracts. If it's possible that the return (percentage profit) on A will exceed the return on B , but not possible that B 's return be larger than A , then any rational investor willing to purchase B would prefer to purchase A . Merton (1992) describes A as being *dominant* over B . It is easy to see that if a market contains a dominant security then there exists an arbitrage opportunity.

This logic leads to some obvious, but useful, statements. Suppose V_t is the value of a portfolio at time $t \in [0, \infty)$. Assuming the portfolio is neither dominant nor dominated and

- $V_t = 0$ for some t , then $V_s = 0$ for all $s > t$;
- $V_t > 0$ for some t , then $V_s > 0$ for all $s < t$.

Similar reasoning may be used to justify the inequalities in the following sections.

2.2 Inequalities involving call options

For the remainder of this chapter, let $C(S_t, T - t, K)$ and $c(S_t, T - t, K)$ be the time t values of an American call option and a European call option respectively. Here $K > 0$ is the strike price, $S_t \geq 0$ is the stock price, $T > 0$ is the time of maturity and $T - t \geq 0$ is the time until maturity. Arbitrage arguments may be used to establish some inequalities which any rational pricing theory for these options must satisfy.

Since both of these options have non-negative payoff, their time t values must also be non-negative

$$C(S_t, T - t, K) \geq 0, \quad c(S_t, T - t, K) \geq 0.$$

Indeed, at expiration both contracts are worth

$$C(S_T, 0, K) = c(S_T, 0, K) = (S_T - K)^+. \quad (2.1)$$

If the underlying stock price is $S_t = 0$ then at expiry $S_T = 0$ and the payoffs in (2.1) must be zero also. If the call options are not to be dominated

$$C(0, T - t, K) = c(0, T - t, K) = 0.$$

An American option may be held until the expiry date and exercised as if it was a European option. The current value must therefore be at least that of the European,

$$C(S_t, T - t, K) \geq c(S_t, T - t, K). \quad (2.2)$$

The time t value of an American option with expiry date T_1 is greater than that of an similar option with expiry $T_2 < T_1$,

$$C(S_t, T_1 - t, K) \geq C(S_t, T_2 - t, K), \quad (2.3)$$

since not only can the first option be exercised at any time up to T_2 , but it can also be exercised in the interval $(T_2, T_1]$. If C is differentiable with respect to time, Equation (2.3) implies

$$-\frac{\partial C(s, T - t, K)}{\partial t} = \frac{\partial C(s, T - t, K)}{\partial T} \geq 0.$$

Both the American and European options monotonic in strike price. If $K_2 > K_1$ are the strike prices then, when it's exercised, the payoff of the option with the least strike, $S_t - K_1$, will be better than the payoff $S_t - K_2$ obtained by the other. And everywhere else the payoff of the first option is always at least that of the second. For there to be no dominant portfolios we require

$$C(S_t, T - t, K_1) \geq C(S_t, T - t, K_2),$$

and

$$c(S_t, T - t, K_1) \geq c(S_t, T - t, K_2).$$

Next suppose I have a portfolio containing one European call option, short in one unit of the underlying stock, and $Ke^{-r(T-t)}$ bonds. At time t this portfolio has the value

$$c(S_t, T - t, K) - S_t + Ke^{-r(T-t)}.$$

At time T , after each of these assets and liabilities have evolved, the portfolio is worth

$$(S_T - K)^+ - S_T + K = (K - S_T)^+,$$

which is non-negative in either case. Thus we've restricted the value of a European call option to be

$$c(S_t, T - t, K) \geq (S_t - Ke^{-r(T-t)})^+, \quad (2.4)$$

which leads us to the following astonishing result.

Theorem 1. *An American call option will never be exercised prior to expiration, and hence it has the same value as its European counterpart.*

Proof. If the option is exercised the holder receives $(S_t - K)^+$. The option will only be cashed in if $(S_t - K)^+ \geq C(S_t, T - t, K)$. But from (2.4), and (2.2), at times prior to expiry ($T - t > 0$), if $S_t > K$ then

$$C(S_t, T - t, K) \geq c(S_t, T - t, K) \geq (S_t - Ke^{-r(T-t)})^+ > (S_t - K)^+.$$

thus the option is always worth more active than dead. \square

2.3 Inequalities involving put options

In light of Theorem 1 and the put-call parity, the value of a European call option uniquely determines the prices of both the corresponding American call option and the European put option. It does not, however, determine the value of the American put. As we shall see, the American put is much harder to price than the others.

In Section 1.1.2 we derived the put-call parity equation for a market which didn't pay interest on loans. The proof centred around the payoff functions for each of the European options. Specifically, we saw that the payoff function for an investor who had bought one call option and sold one European put was

$$c(S_T, 0, K) - p(S_T, 0, K) = S_T - K.$$

Here, consistent with the notation of the previous section, we are denoting the value of European and American put options as $p(S_t, T-t, K)$ and $P(S_t, T-t, K)$ respectively. Without realising it, we then used the arbitrage argument

$$V_T = 0 \implies V_t = 0, \quad \forall t < T,$$

to deduce relationship (1.1) between the call and put option values which holds for all times before maturity. When interest rates are incorporated into the model Equation (1.1) needs to be modified slightly; the argument remains the same.

Theorem 2. *The values of the call option and corresponding European put option are related via the put-call parity equation,*

$$p(S_t, T-t, K) = c(S_t, T-t, K) - S_t + Ke^{-r(T-t)}, \quad (2.5)$$

Proof. At maturity T a portfolio containing one call option, a written European put, short one unit of stock and with K dollars invested is worth

$$V_T = c(S_T, 0, K) - p(S_T, 0, K) - S_T + K = 0.$$

If this portfolio is not to be dominated nor dominating we must have $V_t = 0$ for all $t < T$ regardless of the stock price. At time t we have

$$V_t = c(S_t, T-t, K) - p(S_t, T-t, K) - S_t + Ke^{-r(T-t)},$$

where $Ke^{-r(T-t)}$ is the amount that needs to be invested at t at the risk-free interest rate r to grow to K at time T . Equation (2.5) follows. \square

A consequence of the put-call parity is that many of the inequalities of the previous section have analogies for the European put option. The put-call parity, however, does not help us locate the correct price for an American put option as Theorem 2 does not apply to these options.

Using arbitrage we can restrict the value of the American put to lie within certain bounds. For instance, since the contract may be exercised at any point in time for a payoff of $(K - S_t)^+$, its value must be at least that. That is to say

$$P(S_t, T - t, K) \geq (K - S_t)^+,$$

Also, since the American option can be held right up until the time of maturity as if it were a European option, we have

$$P(S_t, T - t, K) \geq p(S_t, T - t, K), \quad (2.6)$$

Unlike the call options, where we found $C(S_t, T - t, K) = c(S_t, T - t, K)$ because there was no possibility of early exercise, the American put is often likely to be exercised early. If there is a positive probability of early exercise then a strict inequality holds in equation (2.6) meaning the American put will sell for more than its European counterpart. In fact, similar arguments which lead to Theorem 1 may be used to establish

$$p(S_t, T - t, K) \leq P(S_t, T - t, K) \leq p(S_t, T - t, K) + K(1 - e^{-r(T-t)}),$$

while it is possible to have values of S_t which contradict the inequality $P(S_t, T - t, K) \leq p(S_t, T - t, K)$.

2.4 Exercises

1. Given

$$P(S_t, T - t, K) \leq K - S_t + C(S_t, T - t, K),$$

prove that

$$p(S_t, T - t, K) \leq P(S_t, T - t, K) \leq p(S_t, T - t, K) + K(1 - e^{-r(T-t)}).$$

2. Theorem 8.13 of Merton (1992) states that if it's possible that $c(S_t, T - t, K) < K(1 - e^{-r(T-t)})$ then $p(S_t, T - t, K) < P(S_t, T - t, K)$. Rewrite Merton's proof in your own words or give an alternative justification.

3. Prove

$$P(S_t, T - t, K) \leq K - S_t + C(S_t, T - t, K).$$

4. A share currently trades at \$60. A call option with exercise price \$58 and maturity in three months trades at \$3. The three month risk-free interest rate is 5%. A put is offered on the market, with exercise price \$58 and expiry in three months, for \$1.50. Does there exist an arbitrage opportunity? If so, how could you take advantage of it?