Chapter 4

Hedging in continuous time

In the discrete time binomial asset pricing model of Chapter 3 we hedged an option contract by creating a replicating portfolio (see Example 7). This portfolio constructed in terms of stocks and bonds, perfectly matched the performance of the derivative security. And so, by purchasing the replicating portfolio whenever we sell an option we can avoid any risk of making a loss. This is what financial institutions like; they make their money in the transaction not in the randomness of the market.

It was possible, on our binomial tree, to readjust our portfolio at each time tick. The decision was based on the current stock price and the range of possible future values. In the reality though, prices are not confined to a binomial tree and changes in stock value may occur at any time. While we can pass to the continuous time limit for the value of the option, we cannot trade continuously in our attempts to replicate it. For these reasons the replication strategy fails to be as effective in continuous time. In this section we briefly discuss various hedging schemes which are employed in the real market.

Options and other derivative securities are traded either through an organised market (*exchange*) or in the *over the counter* market. The later requires direct contact between participants (via phones or computers) and contract terms may be anything, subject to negotiation. In an organised exchange market, on the other hand, contracts are standardised and a clearing house stands between the buyer and the seller. Effectively the clearing house guarantees the integrity of every contract acting as the buyer for every seller and seller for every buyer.

Any financial institution that sells an option or other derivative security is faced with the task of hedging its risk. If the contract happens to be the same as one that is sold on an exchange, the financial institution can neutralise its exposure by buying on the exchange the same option as it has sold. However it is often the case that the contract has been tailored to the needs of the client and does not correspond to the standardised products traded on exchanges. In this case hedging the exposure is far more difficult.

4.1 Crude hedging schemes

One strategy open to the writer of an option is to simply do nothing. Upon maturity, the contract is either exercised or ignored. From the writer's point of view it is much better if the option finishes out of the money since their maximum profit will then be realised. By doing nothing, however, the writer risks a potentially large loss if the option finishes deep in the money.

The writing of an option without any offsetting position in an underlying asset is referred to as writing a *naked option*. Writing a call option, while simultaneously owning the underlying stock, is called a *covered call*.

The covered position is the opposite extreme to a naked option. If the contract finishes in the money and the contract is exercised then this strategy performs well. On the other hand, if the contract finishes out of the money then the writer may be exposed to a large loss. Neither a naked position nor a covered position provides a satisfactory hedge.

Example 11 (Covered put). *Consider a portfolio consisting of one sold European put option and the equivalent unit of stock sold short. In setting up this position the writer receives P_0 for the option and s_0 for the stock.*

Upon maturity, if $S_T < K$ then the option is exercised and the writer pays K for the stock which he was originally paid s_0 for. The payoff function in this situation is $s_0 - K$. Conversely, if the stock price is such that the option is out of the money then the option is not exercised and the writer needs to buy stock in the free market *and pass it on to cover the original sale. In summary, the payoff function is*

$$\begin{cases} s_0 - K, & \text{if } S_T < K; \\ s_0 - S_T, & \text{if } S_T > K. \end{cases}$$

The writer of the covered put is at risk of losing an unbounded amount if $S_T >> s_0$.

Another basic scheme is known as the *stop loss strategy.* The objective is to hold a naked position whenever it's favourable to do so, and a covered position otherwise. This is achieved by buying or selling the stock each time the price passes the strike. Applying this strategy to hedge a call option, for instance, would require the institution to be holding stock whenever the call is in the money ($S_t > K$), and sell whenever the call is out of the money.

4.2 More sophisticated schemes

A trader aiming to hedge against an option cannot make continuous adjustments to his portfolio, besides other practical problems the presence of transaction costs prevents it. If the interval between portfolio adjustments is too large then it is unlikely that any hedging strategy will be self-financing. Market movements which are adverse to the portfolio will not be accounted for immediately, and in the interim the portfolio will lose value.

The best we can hope for is to make changes at relatively small intervals δt. For sufficiently small δt, it is hoped that the portfolio will be almost self-financing in the sense that the cost of re-balancing at each time increment will be negligible. In practice though, there is a trade off between the size of δt and the transaction costs incurred. Therefore we discuss a more advanced hedging scheme for our continuous time model, assuming no transaction fees while maintaining that δt may be small but is bounded away from zero.

4.2.1 Delta hedging

Suppose a trader needs to hedge against a option which has been sold. If the option is not of standard type, the trader's task is to decide on a hedging portfolio to purchase against the liability of the option[1] C_t. Let V_t denote the hedging portfolio's value, and so the trader will have $\Pi_t = -C_t + V_t$. The use of Π to denote the value of the entire portfolio is standard in the literature.

Now following the approach of Section 3.2, suppose that the hedging portfolio consists of ϕ_i units of stock and ψ_i in cash bonds. That is to say $V_t = \phi_i S_t + \psi_i$. The portfolio's value V_t, the stock price S_t, and the time value of money B_t, each vary continuously in time. The proportions ϕ_i and ψ_i may only be altered if t is a multiple of δt (here $t = i\,\delta t$). Over the interval $[t, t+\delta t)$, the portfolio changes in value to $V_{t+\delta t} = \phi_i S_{t+\delta t} + \psi_i\left(\frac{B_{t+\delta t}}{B_t}\right)$, with a jump of

$$\Delta V_t = \phi_i \Delta S_t + \psi_i\left(\frac{B_{t+\delta t}}{B_t} - 1\right),$$

where $\Delta S_t = S_{t+\delta t} - S_t$.

The value $C_t = C(t, S_t)$, of the option which we wish to hedge is a function of both the stock price S_t and time t. Using Taylor's expansion for functions of several variables

$$C_{t+\delta t} = C_t + \frac{\partial C_t}{\partial t}\delta t + \frac{\partial C_t}{\partial S_t}\Delta S_t + \frac{\partial^2 C_t}{\partial t \partial S_t}\delta t \Delta S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial t^2}(\delta t)^2 + \frac{1}{2}\frac{\partial^2 C_t}{\partial {S_t}^2}(\Delta S_t)^2 + \ldots,$$

therefore $\Delta C_t = C_{t+\delta t} - C_t$ is given by

$$\frac{\partial C_t}{\partial t}\delta t + \frac{\partial C_t}{\partial S_t}\Delta S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial {S_t}^2}(\Delta S_t)^2 + o(\delta t)$$
$$\approx \frac{\partial C_t}{\partial t}\delta t + \frac{\partial C_t}{\partial S_t}\Delta S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial {S_t}^2}\sigma^2 S_t^2 \delta t + o(\delta t).$$

where we've assumed $(\Delta S_t)^2 \approx \sigma^2 S_t^2 \delta t$ to be a known fact; we shall see later that this is related to something called the quadratic variation of the process.

The primary requirement of a hedging portfolio V_t is that it synthesise the value of the option C_t in the next short period $[t, t+\delta t)$. Therefore the trader sets $\Delta V_t = \Delta C_t$

[1] Aiming to keep the discussion as general as possible, we won't specify the contract's details. It could be a call or a put or something more exotic. We shall denote the option's value by $C_t = C(t, S_t)$, as if it were a call option

to get

$$\phi_i \Delta S_t + \psi_i \left(\frac{B_{t+\delta t}}{B_t} - 1 \right) = \frac{\partial C_t}{\partial t} \delta t + \frac{\partial C_t}{\partial S_t} \Delta S_t + \frac{1}{2} \frac{\partial^2 C_t}{\partial {S_t}^2} \sigma^2 S_t^2 \delta t + o(\delta t).$$

Equating like terms yields

$$\phi_i \approx \frac{\partial C_t}{\partial S_t}, \tag{4.1}$$

$$\psi_i \approx \frac{B_t}{B_{t+\delta t} - B_t} \left(\frac{\partial C_t}{\partial t} + \frac{1}{2} \frac{\partial^2 C_t}{\partial {S_t}^2} \sigma^2 S_t^2 \right) \delta t. \tag{4.2}$$

In other words, buying a portfolio with ϕ_i units of stock and ψ_i bonds at time $t = i\,\delta t$ is a strategy which would approximately match the performance of the option in the next short period $[t, t + \delta t)$.

If, for example, C_t *was* a call option and we agree with Black-Scholes' formula for the price at time $t = i\,\delta t$, then

$$C_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2),$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[\log\left(\frac{S_t}{K} \right) + \left(r + \frac{1}{2}\sigma^2 \right)(T - t) \right],$$

and

$$d_2 = \frac{1}{\sigma\sqrt{T-t}} \left[\log\left(\frac{S_t}{K} \right) + \left(r - \frac{1}{2}\sigma^2 \right)(T - t) \right].$$

Differentiating with respect to S_t gives ϕ_i:

$$\frac{\partial C_t}{\partial S_t} = \Phi(d_1) + S_t \Phi'(d_1) \frac{\partial d_1}{\partial S_t} - K e^{-r(T-t)} \Phi'(d_2) \frac{\partial d_2}{\partial S_t}.$$

The second and third terms cancel each other out. To see this observe that $d_2 = d_1 - \sigma\sqrt{T-t}$, which implies $\frac{\partial d_1}{\partial S_t} = \frac{\partial d_2}{\partial S_t}$. And also note that $\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$. What remains is

$$\phi_i = \frac{\partial C_t}{\partial S_t} = \Phi(d_1).$$

The quantity $\frac{\partial C_t}{\partial S_t}$ is known as the *delta* of the option. This term is not reserved for options but applies to portfolios in general. The list of derivatives which are given special names includes

$$\begin{array}{ll} \text{delta} & \dfrac{\partial V_t}{\partial S_t}, \\[2em] \text{gamma} & \dfrac{\partial^2 V_t}{\partial {S_t}^2}, \\[2em] \text{theta} & \dfrac{\partial V_t}{\partial t}, \\[2em] \text{vega} & \dfrac{\partial V_t}{\partial \sigma}, \\[2em] \text{rho} & \dfrac{\partial V_t}{\partial r}, \end{array}$$

where V_t denotes the value of an arbitrary portfolio, it may contain a single option or a combination of contracts, stocks, and bonds.

Figure 4.1 is a plot of $\Phi(d_1)$ against S_t for fixed a fixed time $T - t = 6$. It indicates that for large stock prices, relative to K, the fractional holding of stock in the replicating portfolio of a call approaches 1. On the other hand, there is no point in holding stock when relatively low stock prices are observed as there is little chance the option will be exercised. When the stock price is at the money, the stock holding is roughly equal to one half.

We now have a suggested guide as to how to hedge in continuous time. If we write a call option then we should go out and buy the portfolio (ϕ_0, ψ_0) worth $V_0 = \phi_0 s_0 + \psi_0$. After a short period δt the portfolio is worth $\phi_0 S_{\delta t} + \psi_0 B_{\delta t}$. We then re-balance our assets selling our (ϕ_0, ψ_0) portfolio to finance the purchase of ϕ_1 and ψ_1 units of stock and bonds respectively. Here ϕ_1 and ψ_1 are given by Equations (4.1) and (4.2) above. There is no guarantee that the value of the old portfolio and the new are exactly equal,

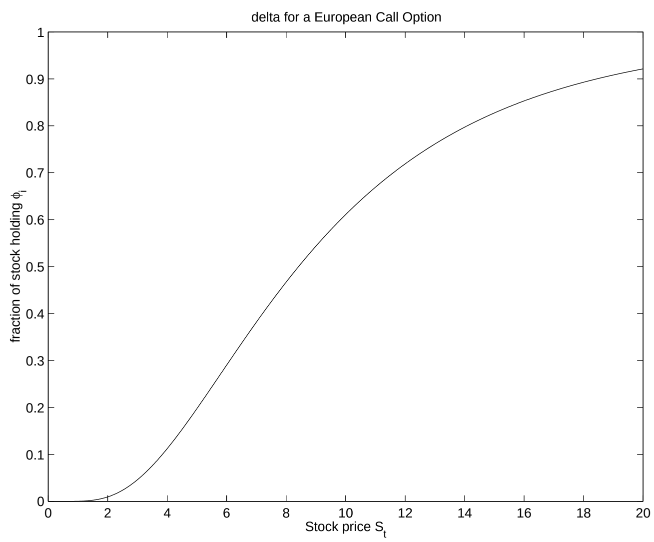

Figure 4.1: delta for a call option with strike $K = 10$, and 6 time units until maturity. The underlying stock process is assumed to have volatility $\sigma = 0.25$, and the risk free interest rate $r = 6\%$.

but for small δt we hope this to be the case. Applying this tactic of re-balancing at each time tick should provide a satisfactory hedge.

4.2.2 Delta-Gamma hedging

The major weakness of the previous strategy that the values of sequential portfolios may not match. For instance, if

$$\phi_i\, S_{t+\delta t} + \psi_i \frac{B_{t+\delta t}}{B_t} < \phi_{i+1}\, S_{t+\delta t} + \psi_{i+1},$$

then extra cash will be required to re-balance and the strategy fails to be self-financing. We cannot guarantee both sides of the above equation to be exactly equal, since it depends on the random value $S_{t+\delta t}$. We can, however, refine our portfolio to have a value which is resistant to small perturbations in the price of the underlying asset. This is known as *delta-gamma hedging*.

Again suppose we wish to hedge against a written option. We can stabilise the hedging portfolio of the previous section by allowing it to contain other contracts in addition to the stocks and bonds. For example, if the contract was an over-the-counter option with non-standard features then we might choose to stack the hedging portfolio with other options on the same stock, but which are available in abundance via the exchange.

Consider then, a portfolio containing ξ_i options each worth F_t, ϕ_i units of stock and ψ_i dollars at time $t = i\,\delta t$. The value of this portfolio is

$$V_t = \xi_i F_t + \phi_i S_t + \psi_i.$$

And the value of the entire portfolio is $\Pi_t = -C_t + V_t$, where C_t represents the value of the (arbitrary) option we wish to hedge.

Again, our first task is to ensure that the hedging portfolio V_t will match the option's value C_t in the next short period. We could repeat the $\Delta V_t = \Delta C_t$ argument of the preceding section, however, for models with the stock price not restricted to a lattice, it is easier (and equivalent) to set $\frac{\partial \Pi_t}{\partial S_t} = 0$.

Differentiating with respect to S_t yields

$$\begin{aligned}\frac{\partial \Pi_t}{\partial S_t} &= -\frac{\partial C_t}{\partial S_t} + \frac{\partial V_t}{\partial S_t}, \\ &= -\frac{\partial C_t}{\partial S_t} + \xi_i \frac{\partial F_t}{\partial S_t} + \phi_i.\end{aligned}$$

Now if $\frac{\partial C_t}{\partial S_t}$ is large then a change in the underlying S_t will bring about a rapid change in the value of the contract C_t. The hedging portfolio V_t will match this change for small variations of S_t, but for large stock movements ΔS_t it is unlikely that $V(t, S_t + \Delta S_t)$ and $C(t, S_t + \Delta S_t)$ will remain equal. The option's value C_t is typically non-linear, and even though the hedging portfolio is chosen such that $V_t = C_t$ and $\frac{\partial V_t}{\partial S_t} = \frac{\partial C_t}{\partial S_t}$, V_t may not match C_t in convexity. Thus, if when the stock price has shifted significantly the values and rates of change of C_t and V_t are likely to be different.

Figure 4.2 illustrates this point. The solid line in each of the panes is the value of a call option C_t with strike price \$20 for times 15, 10, 5, and 0 days from maturity (left to right then top to bottom). The dashed line is the value of a portfolio $V_t = \phi_i S_t + \psi_i$, balanced to hedge the call option 15 days from maturity when the stock price was \$20. In the first pane, the hedging portfolio is a tangent to the value of the call, intersecting at $S_0 = 20$. This means that the hedging portfolio initially matches the value of the call. Also $C_t \approx V_t$ even after small instantaneous stock movements.

If the hedging portfolio is not readjusted frequently, its value may not match that of the call. The second and third panes are for the times 5 and 10 days since the portfolio was re-balanced. Five days since the re-balance, if the stock price has moved more than about 10% either direction away from the initial price \$20, then the hedging portfolio will not be worth as much as the call. Ten days out, and the gap has increased to roughly 13%[2]. If the stock price deviates a long way from the initial value, and the portfolio is not re-balanced in the mean time, then it will fail to provide an adequate hedge.

Sometimes the difference between C_t and V_t will increase the value of $\Pi_t = -C_t + V_t$, other times it will cause value of the entire portfolio to drop. From a bank's perspective this uncertainty is undesirable.

[2] Of course, these rough percentages depend on the assumed volatility and the interest rate (which I haven't quoted because they are unrealistic).

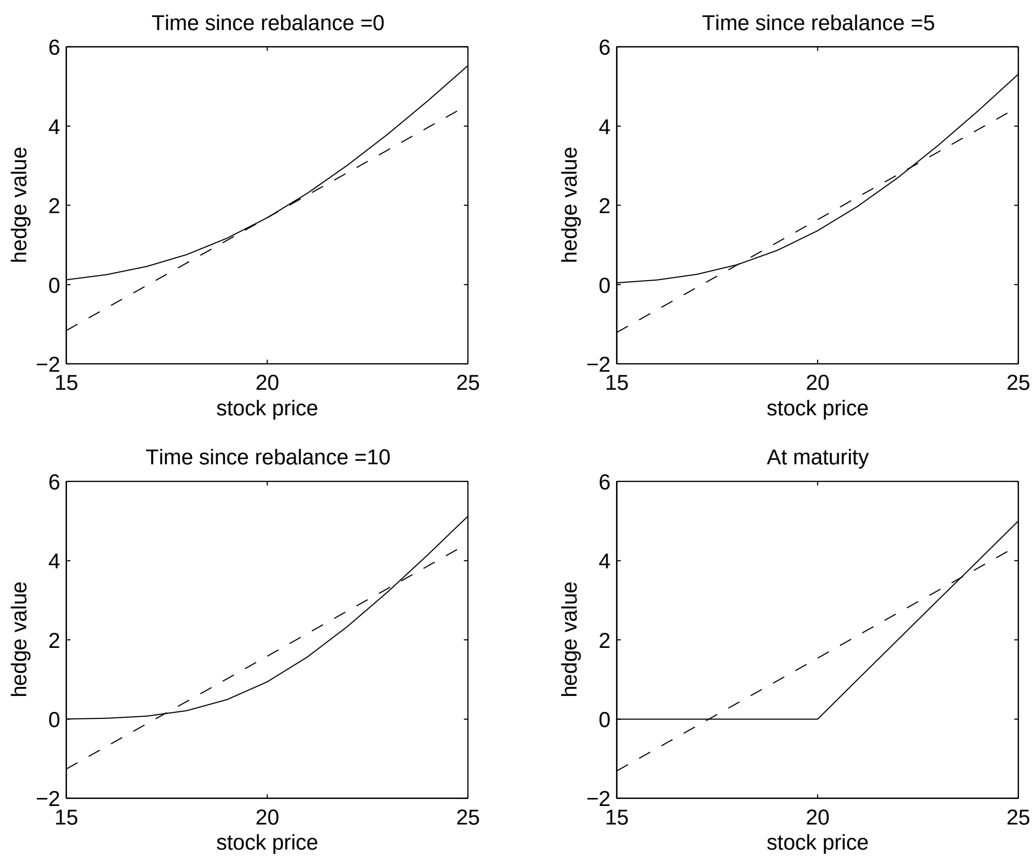

Figure 4.2: A comparison of the value of a call option and the hedging portfolio's value (dashed line) for various times.

So we shall attempt to make the portfolio less sensitive to stock price changes by incorporating ξ_i units of another contract into our portfolio. Choosing (ξ_i, ϕ_i, ψ_i) such that changes in the underlying don't impact quite so heavily on the size of the difference between C_t and V_t.

The delta hedging scheme can be summarised as follows: at each re-balance $t = i\,\delta t$ choose (ϕ_i, ψ_i) such that $V_t = \phi_i S_t + \psi_i$ satisfies

$$\begin{cases} \Pi_t = -C_t + V_t = 0, & \text{self-financing;} \\ \frac{\partial \Pi_t}{\partial S_t} = 0, & \text{delta neutral.} \end{cases}$$

The first condition is that the hedging portfolio costs the value of the call to set up. The second is that the deltas of C_t and V_t also match. When the second condition holds, small changes in the underlying should not interfere with the first condition.

We can further stabilise the portfolio by introducing a third condition which will slow the rate at which the second condition ceases to be approximately holding. Recall that $gamma = \frac{\partial delta}{\partial S_t}$. If gamma is small then delta changes only slowly and adjustments to keep a portfolio delta neutral need only be made infrequently. This suggests we choose (ξ_i, ϕ_i, ψ_i) to satisfy the following system of equations

$$\begin{cases} \Pi_t = 0, & \text{self-financing;} \\ \frac{\partial \Pi_t}{\partial S_t} = 0, & \text{delta neutral;} \\ \frac{\partial^2 \Pi_t}{\partial {S_t}^2} = 0, & \text{gamma neutral.} \end{cases}$$

Further refinements include reducing the sensitivity to changes in volatility; *delta-gamma-vega hedging*.

4.3 Exercises

The Matlab files *call.m*, *delta.m*, *hedgepf.m*, and *plots.m* available at

`www.maths.uq.edu.au/~mrt/ms479/`

may help with these exercises.

1. What drawbacks do you think the stop-loss strategy has?

2. Plot the delta of a European put option and use it to complete the following statements:

   (a) When the stock price is small relative to the strike, the hedging portfolio should contain ____________ stock. This corresponds to the option being ____________ the money, and hence ____________ of stock is required to counter the sold put.

   (b) When the stock price is large relative to the strike, the hedging portfolio should contain ____________ stock. This corresponds to the option being ____________ the money, and hence ____________ of stock is required to counter the sold put.

   (c) When the stock price is close to the strike, the hedging portfolio should contain ____________ of stock and ____________ in cash.

3. Plot the delta for a butterfly spread, as described by Figure 1.5. Describe the features and interpret these in terms of a hedging strategy.

4. Suppose we have a portfolio containing one sold European put option P_2 with strike K_2 and bought one European put option P_1 with strike $K_1 < K_2$. Specify a portfolio (ϕ, ψ) which will delta hedge our exposure.

5. Under the change of variables $G_t := C_t - \xi_i F_t$ the delta-gamma hedging scheme almost looks like the plain delta hedging scheme. What is the extra condition, that is, how is ξ_i chosen?

6. Suppose I wish to delta-gamma hedge a call option C_t which has a non-standard expiry date T_c. Assume there are other call options F_t on the same stock available through the exchange, which have the same strike K but which have maturity date T. Show that ξ_i should be chosen at $t = i\,\delta t$ as

$$\sqrt{\frac{T-t}{T_c-t}}\, e^{-\frac{1}{2}\left(d_c^2 - d_f^2\right)},$$

   where

$$d_c = \frac{1}{\sigma\sqrt{T_c - t}}\left[\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T_c - t)\right],$$

and

$$d_f = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right].$$

7. Suppose I wish to delta-gamma hedge a call option C_t which has a non-standard strike K_c. Assume there are other call options F_t on the same stock available through the exchange, which have the same maturity T but which have strike K. Show that ξ_i should be chosen at $t = i\,\delta t$ as

$$e^{-\frac{1}{2}\left(d_c^2 - d_f^2\right)},$$

where

$$d_c = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{S_t}{K_c}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right],$$

and

$$d_f = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right].$$

8. Suppose C_t and F_t to be call options with different strikes on the same stock. Plot the delta of a portfolio $G_t = C_t - \xi\, F_t$ against S_t and ξ. Find the minimum delta of G_t. Does this correspond to the gamma of G_t equalling zero? How do you interpret your finding?

9. Create graphs for the European put option which are analogous to those in Figure 4.2. Is the unbalanced hedging portfolio worth more than the option after sudden stock shifts or after long periods of very little movement?