Questions for quantum chaos section.

October 22, 2007

Q4 What is the BGS conjecture ? In less than one page explain what is has to say about the quantum description of systems that are classically chaotic. (10 marks)

In 1984, Bohigas, Giannoni and Schmidt (BGS) made the conjecture that:

> "statistical properties of long sequences of energy levels of generic quantum systems whose classical counterparts are chaotic have their pattern in long sequences of eigenvalues of large random Hermitian matrices with independent, identically distributed entries."

The conjecture refers to the level spacing distribution, $P(s)$. This is the probability that the energy level spacing between a randomly chosen level and the next lies between s and $s + ds$. To facilitate comparisons between different systems, the Hamiltonian is scaled to ensure that the average level spacing is unity. For integrable systems the spacing distribution is Poisson distribution. This corresponds to the spectra of fully integrable systems. In such systems the hamiltonian is completely diagonalised and each eigenvalue makes up a symmetry class of its own. It is thus reasonable to assume the eigenvalues are completely uncorrelated. Let the probability to find an eigenvalue between E and $E + dE$ be a constant which we take as unity after rescaling the average energy eigenvlaue spacing. Let us now calculate the probability, $p(s)$, that from a given eigenvalue, we will find only one other a distance s away, with no other eigenvalues between. If we divide up the interval of length s into N equally spaced intervals, the probability is easily calculated as

$$p(s)ds = N \overset{lim}{\to} \infty \left(1 - \frac{s}{N}\right)^N ds \tag{1}$$

In the limit this is $p(s) = e^{-s}$. We note that this s peaked at zero spacing indicating that levels tend to come in bunches.

We can classify hermitian matrices into classes according to how they transform under different classes of unitary transformations. If the Hamiltonian is not time reversal invariant (eg it contains a magnetic field), then the most we can require is that the hermiticity property is preserved. The ensemble of such random matrices is then called the *random unitary ensemble.* This defines the Gaussian Unitary ensemble (GUE).

If the Hamiltonian is invariant under time reversal and does not contain spin half interactions, it can always be chosen as real. This property is preserved under orthogonal transformations.

$$H' = OHO^T \qquad OO^T = 1 \tag{2}$$

In that case we must restrict the class of unitary transformations to be simply orthogonal matrices, and we will refer to the ensemble of random matrices with this property as an *orthogonal ensemble.* This defines the Gaussian orthogonal ensemble (GOE)

In systems with time reversal invariance and with spin interactions Hamiltonians are even dimensional and transform into each other under the *symplectic* transformation:

$$H' = SHS^R \tag{3}$$

where S is a sympletic matrix which means it must satisfy $SS^R = 1$ with

$$S^R = ZS^TZ^{-1} \tag{4}$$

where

$$Z_{nm} = i\delta_{nm}\sigma_y \tag{5}$$

This defines the Gaussian Symplectic ensemble (GSE).

The eigenvalue spacing distributions for the GUE and the GSE as well as the GOE.

$$p(s) = \begin{cases} \frac{\pi}{2} s \exp\left(-\frac{\pi}{4}s^2\right) & \text{(GOE)} \\ \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi}s^2\right) & \text{(GUE)} \\ \frac{2^{18}}{3^6\pi^3} s^4 \exp\left(-\frac{\pi}{4}s^2\right) & \text{(GSE)} \end{cases} \tag{6}$$

Note the dependence at small spacing. The GOE is linear, the GUE is quadratic and the GSE is quartic.

Q5 Consider a system with integrable (regular) classical dynamics. Suppose the corresponding quantum system is described by an N dimensional Hilbert space. Assume that the dynamics of this system is well described by a suitable random matrix ensemble and order the energy levels so that $\epsilon_1 \leq \epsilon_2 \ldots \leq \epsilon_N$. Let $p_r(\lambda)$ be the probability of obtaining an energy level spacing of λ for the r^{th}-nearest neighbour spacing (that is to say the energy level spacing $\epsilon_{n+r} - \epsilon_n = \lambda$ for arbitrary n) . Show that for systems with a regular dynamics this is given by

$$p_r(\lambda) = \frac{\lambda^{r-1}}{(r-1)!}e^{-\lambda}$$

assuming we have scaled the energy to give an average level spacing of unity. (10 marks)

The energy levels for integrable systems are statistically independent and follow a Poisson distribution. This means that the probablity per $d\lambda$ to find a level at the end of the interval $[\lambda, \lambda+d\lambda)$ is constant, which we take to be γ. At the end we will let $\gamma = 1$ to have unit average spacing. Define a particular sample for r levels on the interval $[0, \lambda)$ to be $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$. The probability for this sample is

$$p_r(\lambda|\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r) = e^{-\gamma(\lambda-\lambda_r)}\gamma e^{-\gamma(\lambda_r-\lambda_{r-1})}\gamma \ldots \gamma e^{-\gamma\lambda_1} d\lambda_r \lambda_{r-1} \ldots d\lambda_1$$

We now need to average over all samples on the interval $[0, \lambda)$

$$p_r(\lambda) = (\gamma)^{r-1} \int_0^\lambda d\lambda_r \int_0^{\lambda_r} d\lambda_{r-1} \ldots \int_0^{\lambda_2} d\lambda_1 e^{-\gamma(\lambda-\lambda_r)} e^{-\gamma(\lambda_r-\lambda_{r-1})} \ldots e^{-\gamma\lambda_1}$$

We now do the λ -ordered integrals integrals one after the other to get

$$p_r(\lambda) = \frac{(\gamma\lambda)^{r-1}}{(r-1)!} e^{-\gamma\lambda}$$

Finally we set $\gamma = 1$ to ensure the average nearest neighbour spacing is unity.

Q6 Consider a quantum system described by a Hilbert space of dimension N. Let $|\psi_0\rangle$ be an arbitrary initial state. The survival probability, $P_{\psi_0}(t)$, is defined as the probability that the system will still be found in its initial state, $|\psi_0\rangle$, after a time $t > 0$. If we average over all initial states and over a suitable random matrix ensemble it is possible to write the average survival probability as

$$\overline{P(t)} = \frac{2}{N+1} + \frac{N-1}{N+1} \int_0^\infty P^N(\lambda) \cos(\lambda t) d\lambda$$

where $P^N(\lambda)$ is the probability that an energy level spacing of λ occurs between any pair of energy levels.

(a) Show that, in general,

$$P^N(\lambda) = \frac{2}{N(N-1)} \sum_{r=1}^{N-1} (N-r) p_r(\lambda)$$

where $p_r(\lambda)$ is the probability of obtaining an energy level spacing of λ for the r^{th}-nearest neighbour spacing (2 marks)

If there are r levels in a list, then there are

- $N-1$ nearest neighbour spacings of width λ, which occurs with probability $p_1(\lambda)$
- $(N-2)$ next-nearest neighbour spacings of width λ with probability $p_2(\lambda)$
- ...
- $N-r$ r th nearest-neighbur spacings of width λ with probability $p_r(\lambda)$.

There are N(N-1)/2 elements in this list and each is equally likely. Thus the total probability for any pair of levels to have separation λ is

$$P^N(\lambda) = \frac{2}{N(N-1)} \sum_{r=1}^{N-1} (N-r) p_r(\lambda)$$

(b) Prove that, for a system with regular dynamics the average survival probability, $\overline{P(t)}$, never drops below its long time limit of $2/(N+1)$, while for chaotic systems it may drop below the long time limit for some time.

(8 marks)

We first note that as $t \to \infty$, the oscillatory part oscillates more and more rapidly and thus we expect that this integral will average to zero. So the long time limit is just $2/(N+1)$.

Given that $P^N(\lambda)$ is well behaved, which is to be expected as it is a probability distribution, it can be seen that as t increases the cosine function oscillates more and more rapidly with λ, and hence the integral approaches zero. For any well behaved $P^N(\lambda)$, the long time limit of the survival probability function averaged over initial conditions and an ensemble of hamiltonians is

$$\langle\langle P(\infty)\rangle\rangle = \frac{2}{N+1} \tag{3.32}$$

To observe the behaviour of the survival probability function relative to this limit, we must examine the integral

$$P^N(t) = \int_0^\infty P^N(\lambda)\cos(\lambda t)d\lambda \tag{3.33}$$

which is a fourier transform of $P^N(\lambda)$. Clearly, if $P^N(t)$ is greater than zero, then the survival probability function is greater than its long time limit, while if it drops below zero, the survival probability function is less than its long time limit.

For the case of a classically regular system, it has been previously noted that the statistics of the energy levels are poissonian, and hence we can write the rth nearest neighbour distributions as

$$p_r(\lambda) = \left[\frac{\lambda\hbar}{\langle\Delta E\rangle}\right]^{r-1} \frac{\hbar\exp[-\lambda\hbar/\langle\Delta E\rangle]}{\langle\Delta E\rangle(r-1)!} \tag{3.34}$$

where $\langle\Delta E\rangle$ is the mean energy level separation.

Using this value to determine $P^N(\lambda)$, one can show that

$$\frac{d^2P^N(\lambda)}{d\lambda^2} \geq 0 \qquad \frac{dP^N(\lambda)}{d\lambda} < 0 \quad for \quad \lambda > 0 \tag{3.35}$$

Hence $dP^N(\lambda)/d\lambda$ is non-decreasing, but always remains negative.

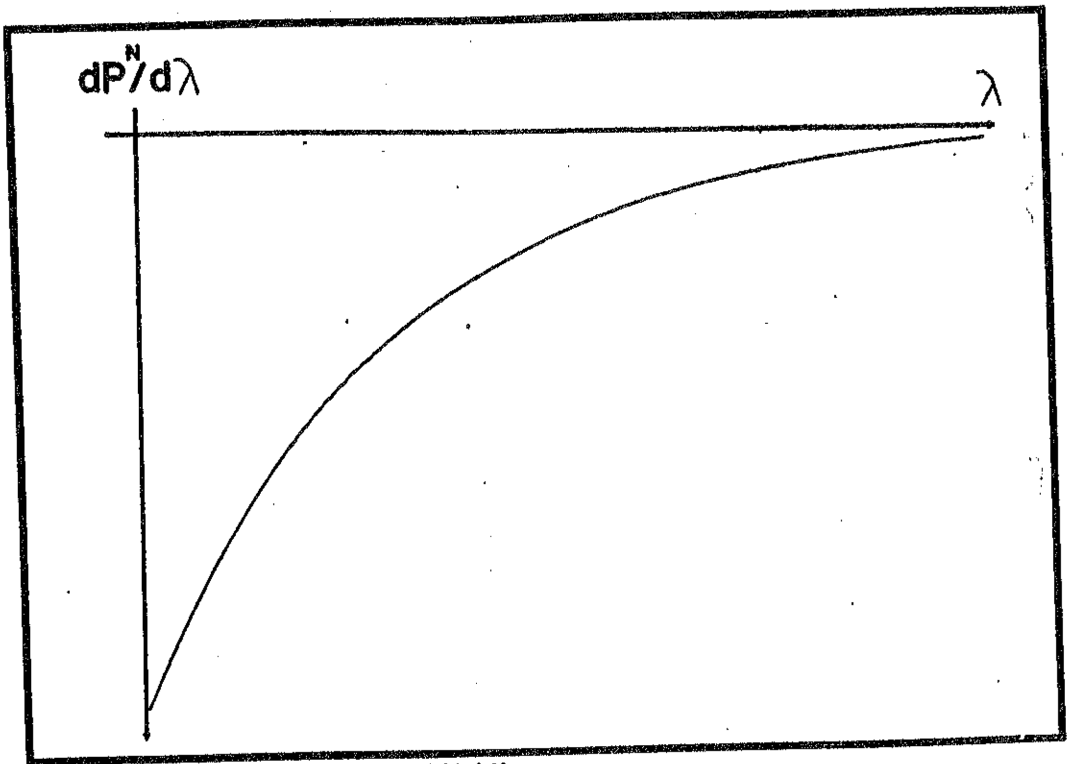

Figure 3.1 Qualitative plot of $dP^N(\lambda)/d\lambda$

Integrating $P^N(t)$ by parts, we obtain

$$P^N(t) = \left[P^N(\lambda) \frac{\sin(\lambda t)}{t} \right]_0^\infty - \int_0^\infty \frac{dP^N(\lambda)}{d\lambda} \frac{\sin(\lambda t)}{t} d\lambda \tag{3.36}$$

where the first term is zero as sin(0)=0 and the probability distribution approaches zero as λ becomes large. We can write the second term as an infinite sum of integrals A_n with limits $2(n-1)\pi/t$ and $2n\pi/t$.

$$\begin{aligned} P^N(t) &= \int_0^{2\pi/t} -\frac{dP^N(\lambda)}{d\lambda}\frac{\sin(\lambda t)}{t}d\lambda \quad +\ldots+ \int_{2(n-1)\pi/t}^{2n\pi/t} -\frac{dP^N(\lambda)}{d\lambda}\frac{\sin(\lambda t)}{t}d\lambda \quad +\ldots \\ &= A_1(t)+\ldots+A_n(t)+\ldots \\ &= \sum_{n=1}^{\infty} A_n(t) \end{aligned} \tag{3.37}$$

Each of the integrals $A_n(t)$ can be subdivided into two terms

$$P^N(t) = \int_{2(n-1)\pi/t}^{(2n-1)\pi/t} -\frac{dP^N(\lambda)}{d\lambda}\frac{\sin(\lambda t)}{t}d\lambda \; + \int_{(2n-1)\pi/t}^{2n\pi/t} -\frac{dP^N(\lambda)}{d\lambda}\frac{\sin(\lambda t)}{t}d\lambda \tag{3.38}$$

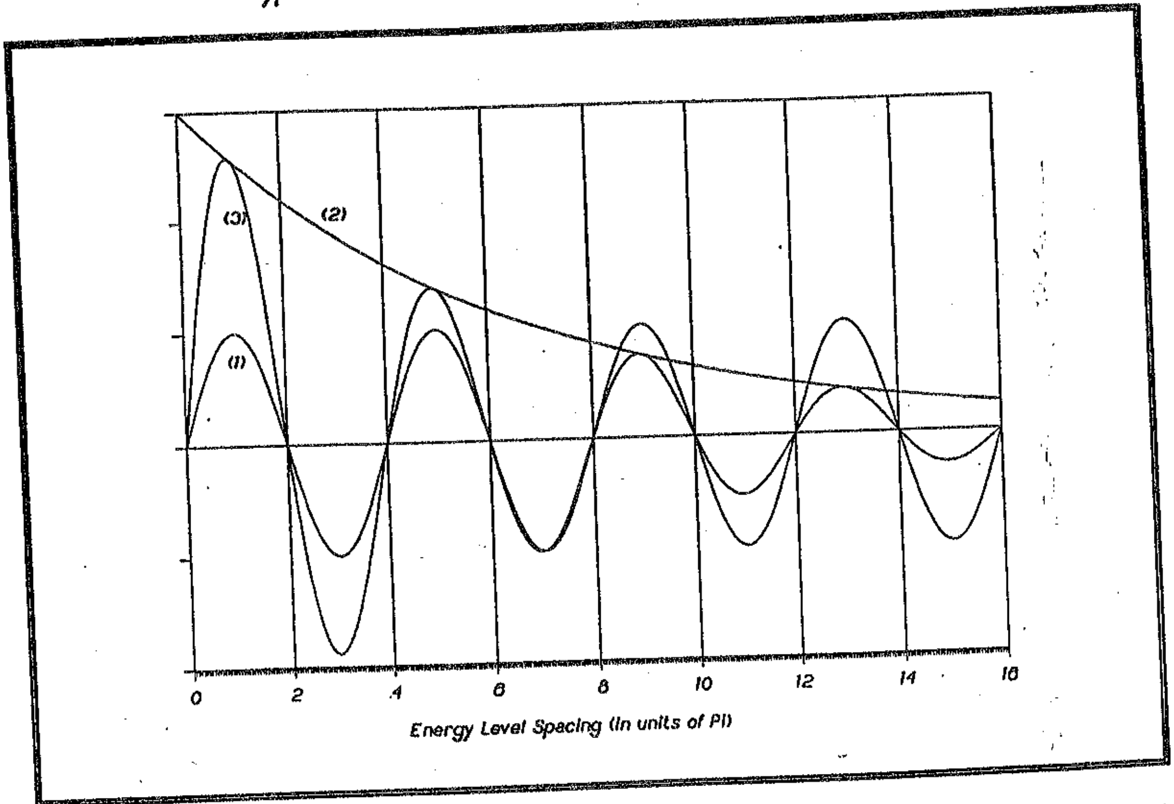

Figure 3.2 Qualitative Plot of (1)sin(λt)/t, -dP^N(λ)/dλ and (3) the multiplication of (1) and (2). The required comparison between compared regions can be clearly seen via the size of the areas.

Now $-dP^N(\lambda)/d\lambda$ is always positive, and $\sin(\lambda t)/t$ is always positive for the first term, and always negative for the second, so the first term will be positive, and the second negative for all values of n. Further, as $-dP^N(\lambda)/d\lambda$ is non-increasing, the magnitude of the first term must be greater than or equal to the magnitude of the second, and thus each of the A_n values must be greater than or equal to zero, for all values of t.

The $P^N(t)$ value must similarly be greater than or equal to zero, hence the survival probability function averaged over initial conditions and an ensemble of hamiltonians must be greater than its long time limit for all times t for a system that is regular in its classical limit.

3.5 Classically Chaotic System

As was established in section 2.3.4, systems which in their classical limit act chaotically are characterised by level repulsion. This is evidenced by the fact that the energy level spacing distribution has a minimum for zero energy spacings. It is also obvious that each of the rth nearest neighbour spacing distributions for $r \geq 2$, must have a minimum at zero, as such an event occurring would require an energy level degeneracy of (r+1) energy levels, which is less likely than a degeneracy with only two energy levels. As $P^N(\lambda)$ is simply the sum of a positive coefficient multiplied by the rth nearest neighbour spacing distributions (see equation 3.24), and each of the rth nearest neighbour spacing distributions must have a minimum at zero energy level spacing, it can be seen that $P^N(\lambda)$ must similarly have a minimum at zero energy level spacing. We can then write

$$\exists\lambda \neq 0 \quad \text{such that} \quad P^N(0) < P^N(\lambda) \tag{3.39}$$

Noting that $P^N(t)$ is the fourier transform of $P^N(\lambda)$, and obviously then that $P^N(\lambda)$ is the reverse fourier transform of $P^N(t)$,

$$\begin{aligned} P^N(0) - P^N(\lambda) &= \frac{2}{\pi}\int_0^\infty dt\, P^N(t)\cos(0t) - \frac{2}{\pi}\int_0^\infty dt\, P^N(t)\cos(\lambda t) \\ &= \frac{2}{\pi}\int_0^\infty dt\, P^N(t)[1-\cos(\lambda t)] \end{aligned} \tag{3.40}$$

Given $P^N(t)\geq 0$ for all $t\geq 0$, it can be seen that for all $\lambda\neq 0$ the above integral must be greater than or equal to zero, as the second factor in the integrand is always non-negative. So

$$\forall\lambda \neq 0 \qquad P^N(\lambda) < P^N(0) \tag{3.41}$$

Clearly there is a contradiction between statements 3.39 and 3.41, and hence our initial assumption that $P^N(t)\geq 0$ is incorrect. We can write

$$\exists t \quad \text{such that} \quad P^N(t) < 0 \tag{3.42}$$

The second part of the proposition, that for a classically chaotic system the survival probability function averaged over initial conditions and an ensemble of hamiltonians drops below its long time limit is hence proved.