MATH4104: Advanced Hamiltonian Mechanics and Chaos.
Assignment Two.
Hand in to before 5.00pm, Friday November 6.

In this assignment you will investigate the classical and quantum dynamics of a driven nonlinear oscillator. Use whatever numerical algorithms you wish, but make sure you implement ways to validate the code (for example by ensuring the classical dynamics is sympletic, and the quantum dynamics is unitary). Your final submission must include a detailed explanation of your results, and how they were obtained together with a source code listing. If you used third party subroutines, eg NAG, simply quote the source. All figures must be fully annotated, and cited in the text.

In the first assignment you studied the driven non linear system defined by the hamiltonian.

$$H(t) = \frac{1}{1 + \frac{p^2}{2} + \omega_0^2 \frac{q^2}{2}} + \epsilon q^2 \cos(t) \ , \qquad (1)$$

where $0 \leq \epsilon \leq 1$. We have explicitly assumed that units have been chosen so that $(H, q, p, t, \omega_0, \epsilon)$ are dimensionless. I will assume in this assignment that you already have a good understanding of the corresponding classical dynamics of this system.

In order to quantise this model, we replace the canonical variables with operators $(q, p) \to (\hat{q}, \hat{p})$ with commutation relations

$$[\hat{q}, \hat{p}] = i\hbar \qquad (2)$$

where $\hbar$ is a dimensionless Planck constant. It is then convenient to define the raising and lowering operators, $(a^\dagger, a)$ as

$$a = \sqrt{\frac{\omega_0}{2\hbar}} \ \hat{q} + \frac{i}{\sqrt{2\hbar\omega_0}} \ \hat{p} \qquad (3)$$

from which it follows that $[a, a^\dagger] = 1$. Thus the Hamiltonian can be written as

$$H(t) = \frac{1}{1 + \hbar\omega_0(a^\dagger a + 1/2)} + \chi(a + a^\dagger)^2 \cos(t) \qquad (4)$$

where

$$\chi = \frac{\hbar\epsilon}{2\omega_0}$$

We will regard the time dependent term as a perturbation so we require $\omega_0 >> \epsilon$ (equivalently, $\hbar\omega_0 >> \chi$).

The eigenstates of the time independent part are eigenstates of the number operator $a^\dagger a|n\rangle = n|n\rangle$, and are thus equivalent to the eigenstates of the simple harmonic oscillator.

1 Assume the initial state is an oscillator coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

Derive an estimate of the first revival time and show that this is consistent with a numerical integration of the dynamics. To demonstrate the revival, compute the average value of a, as a function of time, and plot it in the complex plane.

2 Find the eigenvalues and eigenvectors of the quantum Floquet operator. You will need to truncate the basis set and thus will need to check that your results are not sensitive to the truncation.

3 Find the mean position and momentum as a function of the stroboscopic time for an initial coherent state with a complex amplitude located inside a classical first order resonance of the equivalent classical problem. You should investigate a range of values of $\hbar$.

4 Compare and contrast the classical and quantum dynamics, particuarly for states initially localised in a regular region of phase space.