1

Dynamical localisation.

1.1 The Kicked rotor

We consider a system composed of a free rotor subject to impulsive torques:

$$H(t) = \frac{L^2}{2} + k\cos\theta \sum_n \delta(t-n) \tag{1.1}$$

(we measure time in units of the kick period). The stroboscopic map is then defined by

$$\theta_{n+1} = \theta_n + L_{n+1} \tag{1.2}$$

$$L_{n+1} = L_n + k\sin\theta_n \tag{1.3}$$

where θ_n, L_n are the values of the phase space variables immediately *after* the n'th kick. For small values of k, the phase space is dominated by mostly regular motion. For $k = k_c = 0.9716\ldots$ the last of the invariant torus is destroyed and globally chaotic motion is evident. At this point the angle variable is changing rapidly and seemingly at random from kick to kick while the angular momentum appears to slowly diffuse. This can bee seen as follows. As soon as $|L_n|$ is of the order of 2π or larger, successive θ are uncorrelated due to the modular nature of the angle variable (that is to say mod 2π). In that case the sign of the $\sin\theta_n$ is essentially a random variable and thus successive increments of L_n are random and the angular momentum begins to look like a random walk.

If we begin with an initial distribution of points in phase space, the apparently random, yet deterministic motion will cause the points to move apart rapidly in angle and more slowly in angular momentum. Let the initial distribution be given by $P(L,\theta;0)$. We will assume that this distribution factorises so that

$$P(L,\theta;0) = \frac{1}{2\pi}P_0(L)\ , \tag{1.4}$$

which is uniformly distributed in angle. The contours of the distribution are bands in the cylindrical phase space of the rotor. We will assume that the distribution is very sharply peaked at $L = L_0$, that is $P_0(L) = \delta(L - L_0)$. Under deterministic dynamics the phase space distribution at n kicks is

$$\begin{aligned} P(L,\theta;n) &= \int_{-\infty}^{\infty} dL' \int_0^{2\pi} d\theta' \delta(L - L_n(L',\theta'))\delta(\theta - \theta_n(L',\theta'))P(L',\theta';0) \\ &= \int_0^{2\pi} \frac{d\theta'}{2\pi} \delta(L - L_n(L_0,\theta'))\delta(\theta - \theta_n(L_0,\theta')) \end{aligned} \tag{1.5}$$

We now change variables by defining the $L = L_0 + \Delta L$. The marginal probability for the change in momentum to be ΔL after n kicks is obtained by integrating over θ,

$$P(\Delta L;n) = \int_0^{2\pi} \frac{d\theta'}{2\pi} \delta(\Delta L - L_n(L_0,\theta') + L_0) \tag{1.6}$$

Using the Fourier transform representation of the delta function this may be written as

$$\begin{aligned} P(\Delta L;n) &= \int_0^{2\pi} \frac{d\theta'}{2\pi} \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{ix(\Delta L - L_n + L_0)} \\ &= \int_0^{2\pi} \frac{d\theta'}{2\pi} \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{ix\Delta L} \prod_{p=0}^{n-1} e^{-ix(L_{p+1} - L_p)} \\ &= \int_0^{2\pi} \frac{d\theta'}{2\pi} \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{ix\Delta L} \prod_{p=0}^{n-1} e^{-ixk\sin\theta_p} \end{aligned} \tag{1.7}$$

where we must keep in mind that $L_n = L_n(L_0,\theta')$ and thus $\theta_k = \theta_k(L_0,\theta')$. We now assume that for large p the values of θ_p are uncorrelated, independent random variables with the same distribution as the original uniform distribution. The integral over θ' can be performed using

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-ixk\sin\theta} = J_0(xk) \ , \tag{1.8}$$

where $J_0(y)$ is a Bessel function. Thus

$$P(\Delta L;n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ix\Delta L} \left[J_0(kx)\right]^n \ . \tag{1.9}$$

Now the Nth power of the Bessel function is a rapidly oscillating function of x, so the integral will be dominated by terms near $x = 0$. For large values of n and small x we can use the approximation

$$\begin{aligned} [J_0(kx)]^n &= e^{n \ln J_0(kx)} \\ &= \exp[-\frac{n}{4}(kx)^2 + \ldots] \end{aligned} \tag{1.10}$$

In the limit $n \to \infty$ the unwritten terms in this equation can be neglected. Finally we need only perform a Fourier transform of a Gaussian in Eq.(1.9),

$$P(\Delta L; n) = \frac{1}{k\sqrt{\pi n}} \exp\left[-\frac{(\Delta L)^2}{k^2 n}\right] \tag{1.11}$$

Thus the variance of the increment ΔL is

$$\overline{(\Delta L)^2} = \frac{k^2}{2} n \tag{1.12}$$

We see that the change in momentum increases linearly in discrete time and is thus a diffusion process with a diffusion constant

$$D = \frac{k^2}{2} \tag{1.13}$$

A better approximation for the integral over the Bessel function can be obtained to give departures from this quadratic dependence.

1.2 Quantum kicked rotor

The Floquet operator for kicked systems is easy to obtain (which is why kicked systems are so popular among theoreticians). Given a periodic Hamiltonian of the form

$$H(t) = H_0 + V_0 \sum_n \delta(t - n\tau) \tag{1.14}$$

where τ is the period and V_0 is some potential function. Now replace the delta a function by a pulse of width $\Delta\tau$ and height $(\Delta\tau)^{-1}$, where we will eventually take the limit of $\Delta\tau \to 0$. The we can write

$$H(t) = \begin{cases} H_0 & n\tau \; < \; t \; < \; (n+1)\tau - \Delta\tau \\ H_0 + \frac{1}{\Delta\tau} V_0 & (n+1)\tau - \Delta\tau \; < \; t \; < \; (n+1)\tau \end{cases} \tag{1.15}$$

The Hamiltonian is piecewise constant in time so we can integrate to get the time evolution operator directly, which gives

$$F = \exp\left[\frac{-i}{\hbar}\left(H_0 + \frac{1}{\Delta\tau} V_0\right) \Delta\tau\right] \exp\left(-\frac{i}{\hbar} H_0 (\tau - \Delta\tau)\right) \; . \tag{1.16}$$

Then in the limit $\Delta\tau \to 0$, we get

$$F = \exp\left(-\frac{i}{\hbar}V_0\right)\exp\left(-\frac{i}{\hbar}H_0\tau\right) \tag{1.17}$$

The Floquet operator for the kicked rotor is

$$F = \exp\left(-\frac{i}{\hbar}k\cos\hat{\theta}\right)\exp\left(-\frac{i}{\hbar}\frac{\tau}{2}\hat{L}^2\right) \tag{1.18}$$

where we have included an explicit dependence on the kick period τ. In the basis which diagonalises θ the angular momentum operator is a generator of displacement, thus

$$\hat{L} = \frac{i}{\hbar}\frac{\partial}{\partial\theta} \tag{1.19}$$

The eigenstates of $\hat{L}$ are simply

$$|n\rangle = \frac{1}{\sqrt{2\pi}}e^{in\theta} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{1.20}$$

The matrix elements of F in this basis are easily determined;

$$\begin{aligned}
F_{nm} &= \langle n|F|m\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\exp\left(-\frac{i}{\hbar}k\cos\theta\right)\exp\left(-\frac{i}{\hbar}\frac{\tau}{2}\hat{L}^2\right)e^{im\theta} \\
&= \exp\left(-\frac{i}{\hbar}\frac{m^2\tau}{2}\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left(-\frac{i}{\hbar}k\cos\theta\right)e^{-i(m-n)\theta}d\theta \\
&= \exp\left(-\frac{i}{\hbar}\frac{m^2\tau}{2}\right)i^{m-n}J_{m-n}(k/\hbar)\ . \qquad (1.21)
\end{aligned}$$

In this representation the Floquet matrix elements fall off rapidly as we move away from the diagonal. This gives a banded structure to the matrix.

We can now consider what happens for an initial state $|\psi_0\rangle$. We can expand such states in the diagonal representation of $\hat{\theta}$,

$$|\psi_0\rangle = \int_0^{2\pi} a(\theta)|\theta\rangle \tag{1.22}$$

In the basis which diagonalises the angular momentum operator $\hat{L}$ this state may be written

$$|\psi_0\rangle = \sum_{n=-\infty}^{\infty} a_n|n\rangle \tag{1.23}$$

where the transformation of the coefficients between the two bases is

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} a(\theta)e^{-in\theta}d\theta \tag{1.24}$$

The variance in $\hat{L}^2$ after n kicks is

$$\langle(\Delta\hat{L})^2\rangle_n = \langle\psi_0|(F^\dagger)^n(\Delta\hat{L})^2F^n|\psi_0\rangle \tag{1.25}$$

where $\Delta\hat{L} = \hat{L} - \langle\hat{L}\rangle$.

In the eigenbasis of $\hat{L}$ this may be written

$$\langle(\Delta\hat{L})^2\rangle_n = \hbar^2\left(\sum_{l,k,k'} l^2(F^n)_{lk}(F^n)_{lk'}a_k a^*_{k'}\right) \tag{1.26}$$

where we have assumed that the initial state is chosen so that $\langle\hat{L}\rangle_n = 0$.

We need to choose an initial state which corresponds to the classical case of a distribution that is uniform in angle, but well localised on a particular angular momentum, L_0, which we take to be zero. The simplest example of such a state is the angular momentum eigenstate $|0\rangle$. (See Eq.(1.24)). For such a state it is easy to prove that the average angular momentum remains zero for all time.

In figure 1.1(a) we plot the variance $(\Delta\hat{L})^2\rangle_n$ versus n for kick strength $k = 5$. Also shown is the variance for the classical case with an equivalent initial distribution in phase space. The classical case undergoes diffusive growth as expected. However the quantum case diffuses for a short time (up to around 50 kicks), and then saturates at an almost constant value after about 1000 kicks. The diffusion is suppressed for the quantum case an the state remains localised: this is *dynamical localisation.* In figure 1.1(b) we present the log of the distribution function $P(L; 1000)$, for both the classical and quantum description $(P(L = \hbar l; 1000) = |\langle l|\psi\rangle_{1000}|^2 = P_s(l)$ at kick 1000. The classical case is quadratic indicating an approach to a Gaussian distribution as expected. However the quantum case is very nearly linear. The quantum distribution at saturation can be fitted to

$$P_s(l) = \frac{1}{l_s}\exp\left[-\frac{2|l|}{l_s}\right] \tag{1.27}$$

where l_s is called the *localistion length.*

The fact that the quantum case saturates is entirely due to the discrete nature of the eigenvalue spectrum of the Floquet operator and its banded structure (ie in the angular momentum basis the off diagonal elements fall off rapidly). If we diagonalise the Floquet operator by means of a unitary transformation U,

$$F_{ln} = \sum_k e^{-i\phi_k}U^*_{kl}U_{kn} \tag{1.28}$$

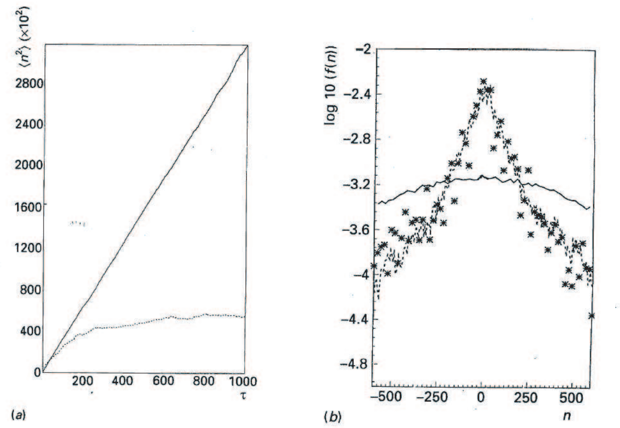

Fig. 1.1. (a) Classical (solid) and quantum (dashed) diffusion for the kicked rotor for $k = 5$. The variance in angular momentum is plotted versus kick number. (b)Classical and quantum marginal distribution fo rnagular momentum after 1000 kicks (See G.Casati and B . Chirikov, *Quantum Chaos*, Cambridge University Press, 1995.)

where ϕ_k are the quasienergies, then for the initial state chosen above

$$\langle \hat{L}^2 \rangle = \hbar^2 \sum_{l,k,k'} l^2 e^{in(\phi_k - \phi_{k'})} U^*_{kl} U_{k0} U_{k'l} U^*_{k'0} \tag{1.29}$$

As the Floquet matrix is banded, so are the unitaries that diagonalise it. Thus the sums in this expression are essentially truncated when the indices k, k' exceed some maximum value l_s. The Floquet phases associated are almost uniformly distributed on the unit circle with a mean density of $l_s/2\pi$ The smallest phase difference occurring in the sum is thus of the order of $2\pi/l_s$. For kick numbers $n \leq l_s$ the system does not see the discreteness of the distribution. For $n >> l_s$ however the phase factors oscillate rapidly and all terms except $k = k'$ average to zero, and

$$\langle \hat{L}^2 \rangle = \hbar^2 \sum_{l,k} l^2 |U_{kl}|^2 |U_{k0}|^2 \tag{1.30}$$

which is independent of the kick number, n. The crossover between the two regions occurs around the kick number $n \sim l_s$ ¿it is also easy to see that when localisation occurs, $\langle \hat{L}^2 \rangle \sim l_s^2$. If we substitute this in the distribution, Eq.(1.27) we find

$$l_s^2 \sim \frac{k^2}{2} l_s = D l_s \tag{1.31}$$

and thus

$$l_s = \alpha D \tag{1.32}$$

which is to say the quantum localisation length is proportional to the classical diffusion constant. Unfortunately the constant of proportionality can only be determined numerically. However for the potential,

$$V(\theta) = V_0 \arctan(\epsilon - 2k\cos\theta) \tag{1.33}$$

(the Lloyd model), the proportionality constant is found analytically to be $\alpha = 1/2$. The numerical results for the kicked rotor indicate that α is also close to $1/2$.

1.3 Experimental demonstration of dynamical localisation.

See handout:

"Atom optics realisation of the quantum δ kicked rotor."

F. L. Moore, J. C. Robinson, C. F. Bharucha, Bala Sundaram, and M. G. Raizen, Phys. Rev. Lett. **75**, 4598 (1995).

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