

Tute Sheet 1, Hand in only the starred questions (Q1b), Q2b) and d), Q4a), Q5, Q7c) and for extra marks Q9).

- 1 For each of the mechanical systems described below, give a configuration space and the number of degrees of freedom of that configuration space.

a) Two juggling pins.

b)* A spherical pendulum, consisting of one point mass hanging from a rigid massless rod attached to a fixed support. The pendulum bob may move in any direction subject to the constant imposed by the rigid rod. The point mass is subject to the uniform force of gravity.

c) A point mass sliding without friction on a rigid curved wire.

Recall the Lagrangian is $L = T - V$ where T is the kinetic energy and V the potential energy. Then the Euler Lagrange equations are:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = 0. \quad (1 \leq j \leq n)$$

- 2 Derive the Euler Lagrange equations of motion for the following systems.

a) A particle of mass, m , moves in a two-dimensional potential $V(x, y) = (x^2 + y^2)/2 + x^2y + y^3/3$, where x and y are rectangular coordinates of the particle.

b*) A particle of mass, m , moves in a two-dimensional potential $V(x, y) = (x^2 + y^2)/2 - x^4/4$, where x and y are rectangular coordinates of the particle.

c) A free particle constrained to move on a sphere of radius R , where the generalized coordinates are the angles θ ; the colatitude of the particle and ϕ the longitude.

d*) A one degree of freedom system with Lagrangian $L(q, \dot{q}) = \frac{1}{2}e^{\alpha t}(\dot{q}^2 - \omega^2 q^2)$.

- 3 Formulate a Lagrangian for a spherical pendulum subject to uniform gravity. What symmetries can you find?

Find coordinates that express that symmetry and analytic expression(s) for the conserved quantity.

- 4 For each of the following Lagrangians derive the Hamiltonian and Hamilton's equations of motion.

a)* $L(t; q, \dot{q}) = \frac{1}{2}e^{\alpha t}(\dot{q}^2 - \omega^2 q^2)$

b) $L(t; \theta, \phi, \alpha, \beta) = \frac{mR^2(\alpha^2 + \beta^2 \sin^2(\theta))}{2}$, where $\alpha = \dot{\theta}$ and $\beta = \dot{\phi}$.

c) $L(x, \dot{x}) = m_0 c^2 \left(1 - \left(1 - \frac{\dot{x}^2}{c^2} \right)^{\frac{1}{2}} \right) - \frac{1}{2} m_0 \Omega^2 x^2$ The relativistic Lagrangian for a particle of rest mass m_0 moving along the x - axis under the simple harmonic potential field $V = \frac{1}{2} m_0 \Omega^2 x^2$.

- 5* Sketch the curves where the Hamiltonian $H = c$, a constant for the system with Lagrangian

$$L(q, \dot{q}) = \frac{\dot{q}^2}{2} + q \left(1 - \frac{q^2}{3} \right)$$

Find the value for c for the solution on the homoclinic orbit. (A homoclinic orbit is a trajectory of a system with a saddle at $\mathbf{x}_0$ such that as $t \rightarrow \infty$ then $\mathbf{x} \rightarrow \mathbf{x}_0$ AND as $t \rightarrow -\infty$ then $\mathbf{x} \rightarrow \mathbf{x}_0$.)

6 The Hamiltonian for Kepler's Central Force Problem is

$$H(r, \theta, p_r, p_\theta) = \frac{p_r^2}{2\mu} - \frac{k}{r} + \frac{p_\theta^2}{2\mu r^2},$$

where $l = r^2\dot{\theta}$.

- a) Write the equations of motion for Kepler's central force problem.
 - b) What symmetries can you find?
 - c) Give analytical expressions for the conserved quantity.
 - d) Using the equations of motion and the conserved quantity derive a second order equation for $r(\theta)$ and solve for r . (It may be useful to let $u = \frac{1}{r}$.)
 - e) $\frac{1}{r} = C(1 + e \cos(\theta - \theta_0))$ describes a hyperbola if $e > 1$ and an ellipse if $0 < e < 1$. Use this to determine for what values of H $r(\theta)$ is a circle, ellipse, parabola or hyperbola.
- 7** For each of the following functions find the function that is related to the given function by the Legendre transform on the indicated active argument. Show that the Legendre transform relations hold for your solution.
- a) $F(x) = ax + bx^2$
 - b) $F(x, y) = a \sin x \cos y$, with x active but y passive.
 - c)* $F(u, x) = \sqrt{(1 - u^2) - x^2}$, with u active but x passive.
- 8** Prove that the Poisson Bracket of two constants of the motion is itself a constant of the motion.

Extra Marks!! can be gained by also solving this problem.

- 9*** Show that the Euler-Lagrange equations of motion for systems with the following Lagrangian's give the same motion:

$$a) \quad L_a(q, \dot{q}, t) = \frac{1}{2}G(q, t)\dot{q}^2 + F(q, t)\dot{q} - V(q, t)$$

$$b) \quad L_b(q, \dot{q}, t) = \frac{1}{2}G(q, t)\dot{q}^2 - \frac{\partial f}{\partial t}(q, t) - V(q, t) \quad \text{where } \frac{\partial f}{\partial q} = F$$

Then find the Hamiltonians for each system. Use this to show that H_a and H_b , given below, give rise to the same motion.

$$H_a(q, p, t) = \frac{(p - \cos(q + t))^2}{2} + \sin(q + t)$$

$$H_b(q, p, t) = \frac{(p)^2}{2} + \sin(q + t) + \cos(q + t)$$