Tute Sheet 3, Hand in only the starred questions.

1* Find the width of the primary resonances for

$$H(\theta_i,\, I_i) = I_1 I_2 + \frac{I_2^2}{2} + \epsilon I_1^2 I_2 \cos 2(2\theta_1 - \theta_2) - \epsilon I_1 I_2 \cos(3\theta_1 - \theta_2)$$

2* Which of the following Hamiltonians are integrable? Give reasons.
If they are not integrable, do KAM tori exist for ϵ small enough?

a) $H(t,\, \theta,\, I) = I + I^2\epsilon(\cos(\theta + \omega t) + \sin^2(\theta + \omega t)$

b) $H(t,\, \theta,\, I) = I^2 + \epsilon(\cos(2\theta + \omega t) + \sin(2\omega t))$

c) $H(t,\, \theta,\, I) = I^2 + \epsilon\cos(\theta)\sin(\theta + \omega t)$

d) $H(\theta_i,\, I_i) = \dfrac{1}{(1 + I_1 I_2)} + \epsilon I_1 \cos(2\theta_1)$

e) $H(\theta_i,\, I_i) = I_1^2 + I_2 + \epsilon I_1^2(\cos(\theta_1 - 3\theta_2)) + \cos(2\theta_1 - 6\theta_2))$

f) $H(\theta_i,\, I_i) = I_1 + I_2 + \cos(5\theta_1 - 2\theta_2) + \cos(\theta_1 + \theta_2)$

g) $H(\theta_i,\, I_i) = I_1 + I_2^2 + \epsilon I_1^{\frac{3}{2}} I_2(\cos(3\theta_1 - 2\theta_2)) + \sqrt{I_1 I_2}\cos(\theta_1 + \theta_2))$

h) $H(\theta_i,\, I_i) = I_1 I_2^2 + \epsilon I_1 I_2(\cos(\theta_1 - 2\theta_2) + \cos\theta_2)$

3 Find continued fraction expansions of the following.

a) $\dfrac{21}{13}$

b) $\dfrac{47 - \sqrt{5}}{38}$

c*) the roots of $x^2 = 2x + 1$.

d) $\dfrac{1 + \sqrt{5}}{2}$ prove the result given in the notes.

4 Find the phase space solutions of a two dimensional linear area preserving map, whose linearized matrix T has trace ± 2.

5 Find the critical points of the Standard Map that lie on the line $\theta = 0 (\text{mod } 2\pi)$ and $\theta = \pi (\text{mod } 2\pi)$ and determine their stability characteristics.

6 Find the first five rational approximates for the noble KAM torus with winding number $\omega = \frac{1}{\gamma^2}$, where γ is the golden mean.

7* a) Suppose I_1 and I_2 are involutions. That is $I_1^2 = I_2^2 = I$ the identity. Prove that if x and $(I_2 I_1)^n x$ are fixed points of I_1 (or I_2) then x is a fixed point of $(I_2 I_1)^{2n}$.

b) Show that the map

$$\begin{aligned} \theta_{n+1} &= \theta_n + g(J_{n+1}) \\ J_{n+1} &= J_n + f(\theta_n) \end{aligned}$$

where $f(-\theta) = -f(\theta)$, is a product of two involutions.

c) Now assuming that $f(\theta) = A\sin(\theta)$ and $g(J) = J^2$ find any lines that are fixed under the evolutions of the map. Using this and part a) investigate the possibility of symmetric period-2 points where one iterate lies on the line $\theta = 0$.

For extra marks!!

8 a) Show that a number with periodic continued fraction expansion satisfies a quadratic equation with integer coefficients.

b) Show that a number with eventually periodic continued fraction expansion satisfies a quadratic equation with integer coefficients.