# MATH4104: Quantum nonlinear dynamics. Lecture TEN. Quantum dynamics in a periodic driven systems.

G J Milburn

The University of Queensland

S2, 2009

# Quantum resonances.

There is a quantum analog of classical second order perturbation theory that leads to the identification of first and second order resonances.

$$\hat{H}(t) = \hat{H}_0 + \epsilon \hat{H}_1(t) \ .$$

For $\epsilon = 0$ the Floquet operator is

$$\hat{F} = \exp\left( \frac{-2\pi i \hat{H}_0}{\hbar} \right) \ ,$$

where $\hat{H}_0 = \hat{p}^2/2 - \kappa \cos \hat{q}$.

Denote the stationary states of $\hat{H}_0$ with energy $E_n(p)$ by $|E_n, p\rangle$, then $|E_n, p\rangle$ is a quasi-stationary state for $\hat{F}$ with quasi-energy $e_n(p) = E_n(p)$.

# Quantum resonances.

In analogy with time-independent quantum perturbation theory we assume that for small $\epsilon$ the perturbed quasistationary states $|e_n, p\rangle$ and quasifrequency $e_n(p)$ are close to $|E_n, p\rangle$ and $E_n(p)$ respectively, and then attempt to find corresponding asymptotic expansions in $\epsilon$.

# Quantum resonances.

Let $|e_n, p, t\rangle$ satisfy the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |e_n, p, t\rangle = \hat{H}(t) |e_n, p, t\rangle$$

subject to the condition $|e_n, p, 2\pi\rangle = \exp(-2\pi i e_n(p)/\hbar) |e_n, p, 0\rangle$.

Define, $|v_n, p\rangle = \exp(i e_n(p) t/\hbar) |e_n, p, t\rangle$ which are periodic in $t$, and

$$e_n(p) |v_n, p\rangle = -i\hbar \frac{d}{dt} |v_n, p\rangle + \hat{H}(t) |v_n, p\rangle \ .$$

# Quantum resonances.

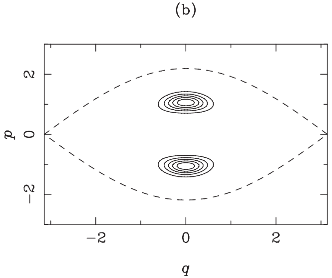
Seek,

$$\begin{aligned} e_n(p) &= E_n(p) + \epsilon e_n^{(1)}(p) + \epsilon^2 e_n^{(2)}(p) + O(\epsilon^3)\ , \\ |v_n, p\rangle &= |E_n, p, 0\rangle + \epsilon |v_n^{(1)}, p\rangle + \epsilon^2 |v_n^{(2)}, p\rangle + O(\epsilon^3)\quad . \end{aligned}$$

As in classical p.t. singularities arise in these expansions at *resonances*.

For each classical resonance $\Delta n\,\omega_{cl}(J) - l = 0$ there will be energy bands with $E_n(p)$ and $E_{n-\Delta n}(p)$ satisfying the near-resonance condition $E_n(p) - E_{n-\Delta n}(p) \approx \hbar l$. The perturbed quasistationary state $|e_n, p\rangle$ will rapidly develop a significant component along $|e_{n-\Delta n}, p\rangle$ as $\epsilon$ is increased.

# Quantum resonances.




This Floquet state is a superposition of a dominant state $|E_{11}, 0\rangle$ and two other states $|E_9, 0\rangle$ and $|E_{13}, 0\rangle$ satisfying the near-resonant conditions: $E_{11}(0) - E_9(0) = 0.103 \approx 2\hbar$ , and $E_{11}(0) - E_{13}(0) = -0.101 \approx -2\hbar$ . The interference between near resonant states has caused the $Q$ function in the figure to become concentrated about the stable regions of the classical second-order resonance.

# Example: kicked rotor.

A free rotor subject to impulsive torques:

$$H(t) = \frac{L^2}{2} + k\cos\theta \sum_n \delta(t-n)$$

The Floquet map is then defined by

$$\begin{aligned} \theta_{n+1} &= \theta_n + L_{n+1} \\ L_{n+1} &= L_n + k\sin\theta_n \end{aligned}$$

Period-one fixed points:

$$\begin{aligned} \theta^* &= \theta^* + L^* \\ L^* &= L^* + k\sin\theta^* \end{aligned}$$

$$L^* = 0 \;\;\&\;\; \theta^* = 0, \pi$$

# Example: kicked rotor.

Stability:

$$\begin{pmatrix} \delta\theta_{n+1} \\ \delta L_{n+1} \end{pmatrix} = \begin{pmatrix} 1 \pm k & 1 \\ \pm k & 1 \end{pmatrix} \begin{pmatrix} \delta\theta_n \\ \delta L_n \end{pmatrix}$$

$$+ : \theta = \pi$$
$$- : \theta = 0$$

1. $\theta = 0$
eigenvalues: $\lambda_\pm = (1 - k/2) \pm i\sqrt{1 - (1 - k/2)^2}$
*stable*: $0 < k < 4$
2. $\theta = \pi$
eigenvalues: $\lambda_\pm = (1 + k/2) \pm \sqrt{(1 + k/2)^2 - 1}$
*unstable*

# Example: kicked rotor.

Area preserving?

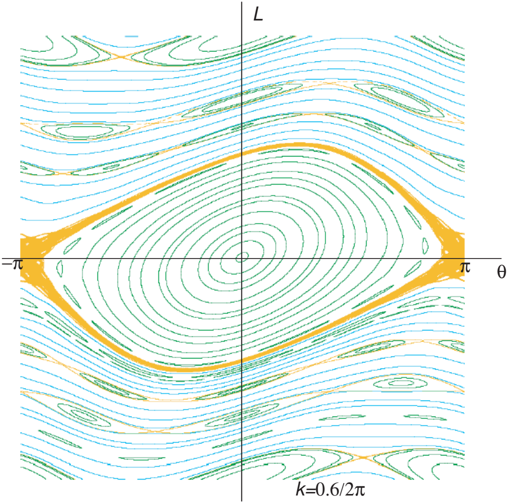
$$\lambda_+\lambda_- = 1$$

Bifurcation?

fixed point at $\theta^* = 0$ undergoes a change of stability for $k > 4$

# Example: kicked rotor.

very rich bifurcation dynamics



# Example: kicked rotor.

Quantum description: unitary Floquet operator, $\hat{F}$, defines a *unitary dynamical map*:

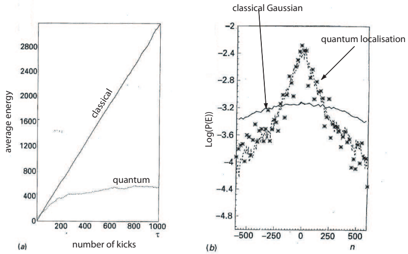
$$|\psi_{n+1}\rangle = \hat{F}|\psi_n\rangle$$

$$\hat{F} = \exp\left(-\frac{i}{\hbar}k\cos\hat{\theta}\right)\exp\left(-\frac{i}{\hbar}\frac{\tau}{2}\hat{L}^2\right)$$

explicit kick period $\tau$.

# Example: kicked rotor.

The classical system shows diffusive growth in energy.




The quantum system does NOT diffuse in momentum!

# Example: kicked rotor.

Experimental verification: cold atoms in optical laser pulses.

VOLUME 75, NUMBER 25 PHYSICAL REVIEW LETTERS 18 DECEMBER 1995

## Atom Optics Realization of the Quantum $\delta$-Kicked Rotor

F. L. Moore,* J. C. Robinson, C. F. Bharucha, Bala Sundaram, and M. G. Raizen

*Department of Physics, The University of Texas at Austin, Austin, Texas 78712-1081*

(Received 21 July 1995)

We report the first direct experimental realization of the quantum $\delta$-kicked rotor. Our system consists of a dilute sample of ultracold sodium atoms in a periodic standing wave of near-resonant light that is pulsed on periodically in time to approximate a series of delta functions. Momentum spread of the atoms increases diffusively with every pulse until the "quantum break time" after which exponentially localized distributions are observed. Quantum resonances are found for specific values of the pulse period.