MATH4104: Quantum nonlinear dynamics.
Lecture ELEVEN.
Quantum Billiards.

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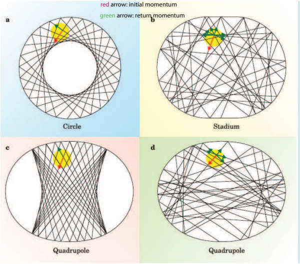
Classical and Quantum billiards.



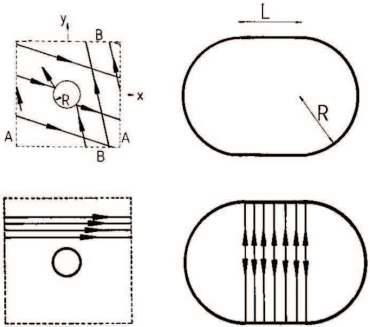
from Stone, Physics Today, August 2005.

ergodic dynamics: all orbits eventually pass through almost every point of the surface of constant energy, $H(p,q) = E$ (a $2N-1$ dimensional surface in general), instead of being confined to an $N-$torus like integrable systems.

Ergodicity is usually associated with chaotic motion (but does not imply it), where we define chaotic motion in terms of sensitive dependance on initial conditions, (eg in terms of a Bernoulli map).

Classical and Quantum billiards.

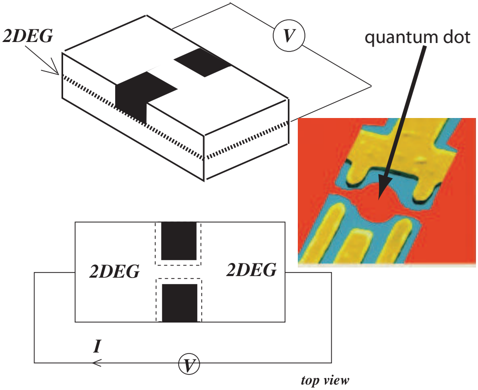
Chaotic billiards: Sinai billiard (left) and Bunimovich stadium (right)



isolated regular orbits (bottom).

Quantum billiards in nanotechnology

The quantum dot...

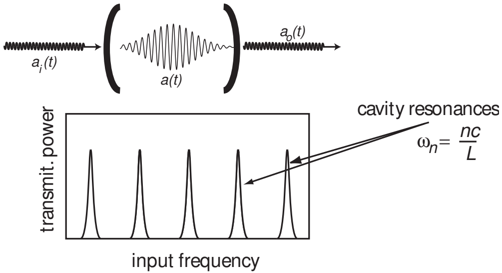


2DEG= two dimensional electron gas.

Quantum billiards in nanotechnology

Conductance depends on matching the incoming electron energy to a resonant energy eigenstate.

An analogy ... matching the frequency of the input light to a cavity transmission spectrum.



Prelude to quantum scattering.

A one dimensional potential,

$$V(x) = \begin{cases} \infty & x \leq 0 \\ 0 & x < a \\ V_0 & a \leq x \leq b \\ 0 & x > b \end{cases}$$

$u_\pm(x)$: position probability amplitude for particles traveling in the $\pm x$ direction, in the region $x > b$, with definite energy $E < V_0$.

These are plane wave states of the form

$$u_\pm(x) = e^{\pm ikx} \tag{1}$$

where $k^2 = \frac{2mE}{\hbar^2}$.

Prelude to quantum scattering.

By matching the boundary condition is it easy to show that at $x = b$,

$$\frac{u_+(b)}{u_-(b)} = -e^{2i\phi(k)}$$

where

$$\tan\phi(k) = \frac{k}{\alpha}\frac{\left[f(k)e^{\alpha(b-a)} + e^{-\alpha(b-a)}\right]}{\left[f(k)e^{\alpha(b-a)} - e^{-\alpha(b-a)}\right]} \tag{2}$$

where

$$\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \tag{3}$$

$$f(k) = \frac{\alpha\tan ka + k}{\alpha\tan ka - k} \tag{4}$$

There is only a phase shift between left-going and right-going states.

Prelude to quantum scattering.

We can calculate the (relative) probability $P(k)$ to find the particle in $0 \leq x \leq a$, relative to the probability to find a particle at $x > b$ with negative momentum,

$$P(k) = 4\left\{\sin\phi(k)\cosh(\alpha(b-a)) - \frac{k}{\alpha}\cos\phi(k)\sinh(\alpha(b-a))\right\}^2 \tag{5}$$

We now define the dimensionless parameters, the scaled energy x, the scaled barrier strength w and the scaled barrier length l,

$$x = ka \tag{6}$$

$$w^2 = \frac{2mV_0}{\hbar^2}(b-a) \tag{7}$$

$$l^2 = \frac{b}{a} - 1 \tag{8}$$

Prelude to quantum scattering.

Plot the function $P(k)$ and $\cos\phi(k)$ versus the scaled input energy, x. This is a kind of spectrum of excitation for the quasi bound state in the well. Note the *resonance* at a particular value of input energy. Why does this maximum occur?

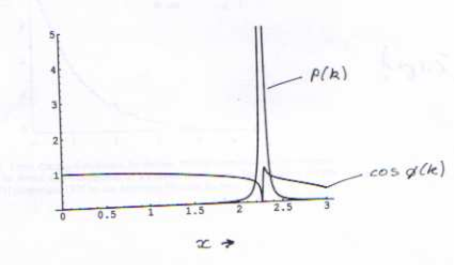


Figure: The relative occupation probability for the bound state versus the input energy with $w = 3, l = 1.0$

Prelude to quantum scattering.

For the infinite square well,

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2ma^2} \quad n = 1, 2, \ldots$$

In the limit of $E << V_0$ the peak of the spectrum corresponds to the allowed energies of the bound state of the infinite square well.

Scattering to billiards.

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To probe the billiard problem spectroscopically we need to weaken the classical perfect billiard system to some extent. We need to keep the disturbance due to this measurement to a minimum.

Given this, we seek to understand the quantum scattering problem by determining the classical billiard problem first.

Quantum billiards.

In a two dimensional billiard system a particle of mass m moves freely in the plane until it encounters a wall, at which point it undergoes a perfectly elastic collision.

Schrödinger equation in the coordinate representation

$$i\hbar \frac{\partial \psi(x,y,t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(x,y,t) \tag{9}$$

subject to the *Dirichelet* boundary condition

$$\psi(x,y,t)|_S = 0 \tag{10}$$

where S denotes the curve that defines the boundary of the billiard.

Quantum billiards.

The energy eigenstates $\psi_n(x,y)$ and allowed energies, $E_n = \hbar\omega_n$, are obtained by separating the time dependence as

$$\psi_n(x,y,t) = \psi_n(x,y)e^{-i\omega_m t} \tag{11}$$

and solving the time independent equation

$$\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\psi_n(x,y) &= E_n\psi_n(x,y) && (12) \\ \nabla^2\psi_n(x,y) + k_n^2\psi(x,y) &= 0 && (13) \end{aligned}$$

with the dispersion relation

$$\omega_n = \frac{\hbar}{2m}k_n^2 \ , \tag{14}$$

subject to the boundary condition.

A microwave analogy for quantum billiards.

Consider an electromagnetic wave sustained inside a conducting boundary; a resonant cavity.

The equation for the electric and magnetic field amplitudes are

$$(\nabla^2 + k^2)\vec{E} = 0 \tag{15}$$

$$(\nabla^2 + k^2)\vec{B} = 0 \tag{16}$$

with the dispersion relation $\omega = ck$ and the boundary conditions

$$\vec{n} \times \vec{E} = 0 \tag{17}$$

$$\vec{n} \cdot \vec{B} = 0 \tag{18}$$

where $\vec{n}$ is a unit normal to the surface of the boundary.

A microwave analogy for quantum billiards.

To make this look like a quantum billiard problem we consider resonators with cylindrical symmetry, but varying cross sections.

Taking the z axis parallel to the axis of cylindrical symmetry the boundary conditions are

$$E_z|_S = 0 \tag{19}$$

$$\vec{\nabla}_\perp B_z|_S = 0 \, , \tag{20}$$

where $\vec{\nabla}_\perp$ denotes the normal derivative.

A microwave analogy for quantum billiards.

These can be satisfied for transverse magnetic modes (TM)

$$E_z(x,y,z) = E(x,y)\cos(n\pi z/d) \quad (21)$$

$$B_z(x,y,z) = 0 \quad (22)$$

where

$$\left[\nabla^2 + k^2 - \left(\frac{n\pi}{d}\right)^2\right] E = 0 \quad (23)$$

with Dirichelet boundary condition $E(x,y)|_S = 0$ on the surface. For frequencies $\nu < c/2d$ ($k < \pi/d$) only TM modes with $n = 0$ are possible.

Thus the electric field in the x, y plane must satisfy $(\nabla^2 + k^2)E = 0$ with Dirichelet boundary conditions. (The dispersion relation for electromagnetic waves however is different: $\omega = ck$).

A microwave analogy for quantum billiards.

The equivalence to the quantum billiard problem is apparent and for this reason many early experiments on quantum billiards were microwave cavity experiments.

A crucial difference: In the microwave case we are concerned with a true field amplitude in the cavity.
In the quantum case the analogous object is a *probability amplitude*.

This means the kinds of measurements that we could consider in the two cases are very different. It is possible to measure a field amplitude, while generally we cannot measure a quantum probability amplitude directly.

A microwave analogy for quantum billiards.

measure the field inside the cavity by introducing a small antenna into the resonator through a small hole, and measuring the amount of power reflected back when the antenna is used to excite the cavity.

The reflected power is monitored as a function of the frequency of the injected signal to give a spectrum.

Each minimum in the reflected power corresponds to a resonant eigenfrequency of the cavity.

A microwave analogy for quantum billiards.

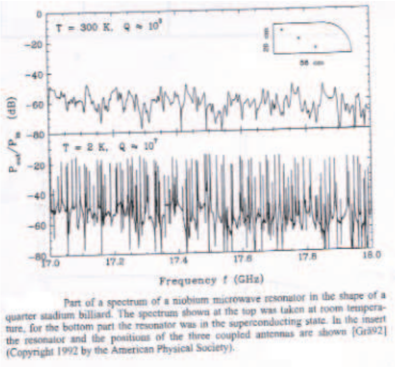

Figure: From 2.12 in Stockman. The spectra at two different temperatures are recorded. At room temperature there is a significant amount of thermally excited background radiation that masks the cavity spectrum. At low temperature with the walls are superconducting, and the spectra reflects more closely the billiard spectrum.

Statistics of spectra.

We will study complex spectra *statistically*.

The sequence of successive eigenvalue differences $s_n = E_n - E_{n-1}$ is computed and the relative frequency of values of s_n plotted to construct a distribution $P(s)$ of eigenvalue spacings.

The variable s_n is scaled so that the average spacing is unity.

FInd that these distributions can be classified into a few general classes that reflect fundamental dynamical properties of the system.

Statistics of spectra.

Eigenvalues either tended to bunch so that $P(s)$ was peaked at zero or they seemed to repel each other so that $P(s)$ is peaked away from zero.

The stadium billiard the spacing distribution is well fitted by the *Wigner distribution*

$$P(s) = \frac{\pi}{2} s \exp[-\frac{\pi}{4} s^2] \tag{24}$$

It is no coincidence that this distribution is associated with a billiard problem that is chaotic.

Statistics of spectra.

If we measure the spectrum of a rectangular cavity, for which the corresponding billiard problem is integrable, we find that

$$P(s) = e^{-s} \tag{25}$$

a Poisson distribution which exhibits level bunching.

The BGS cojecture.

1984 however Bohigas, Giannoni and Schmidt conjectured that:

> *"statistical properties of long sequences of energy levels of generic quantum systems whose classical counterparts are chaotic have their pattern in long sequences of eigenvalues of large random Hermitian matrices with independent, identically distributed entries."*

vindicated in countless real and numerical experiments, yet a rigorous proof is elusive.

Random matrices and number theory.