# MATH4104: Quantum nonlinear dynamics. Lecture TWELVE. Random Matrices.

G J Milburn

The University of Queensland

S2, 2009

# The BGS cojecture.

> *"statistical properties of long sequences of energy levels of generic quantum systems whose classical counterparts are chaotic have their pattern in long sequences of eigenvalues of large random Hermitian matrices with independent, identically distributed entries."*

vindicated in countless real and numerical experiments, yet a rigorous proof is elusive.

# Symmetry.

Let $H$ be Hamiltonian matrix (possibly infinite dimensional)

Let $U$ be the unitary transformation corresponding to some symmetry $U^\dagger H U = H$.

We can then simultaneously diagonalise $U, H$.

Example: hydrogen atom

$$H = \frac{\vec{p}^2}{2m} - \frac{e}{4\pi\epsilon_0 r^2}$$

Invariant under arbitrary rotations of $\theta$ around axis $\vec{n}$

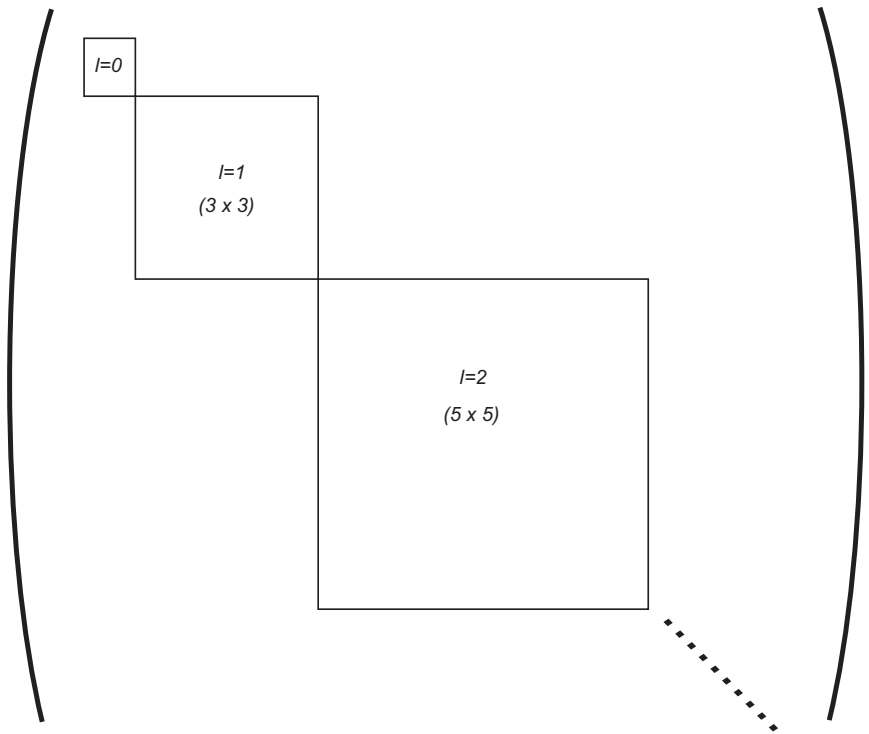
$$U(\vec{n}, \theta) = \exp^{-i\theta \vec{L}\cdot\vec{n}}$$

where $\vec{L}$ is the angular momentum operator.

# Symmetry.

The eigenvalues of total angular momentum are $I(I+1), \ldots I = 0, 1, 2 \ldots$.

We can then *block diagonalise H* so that in each block the angular momentum is fixed



# Random matrices.

Only work with the matrix in a given block, corresponding to one of the eigenvalues of the conserved quantity, and compute eigenvectors and eigenvalues.

For complex systems, the block can be very large and there are many eigenvalues.

Then *rescale the energy* so that the average density is unity.

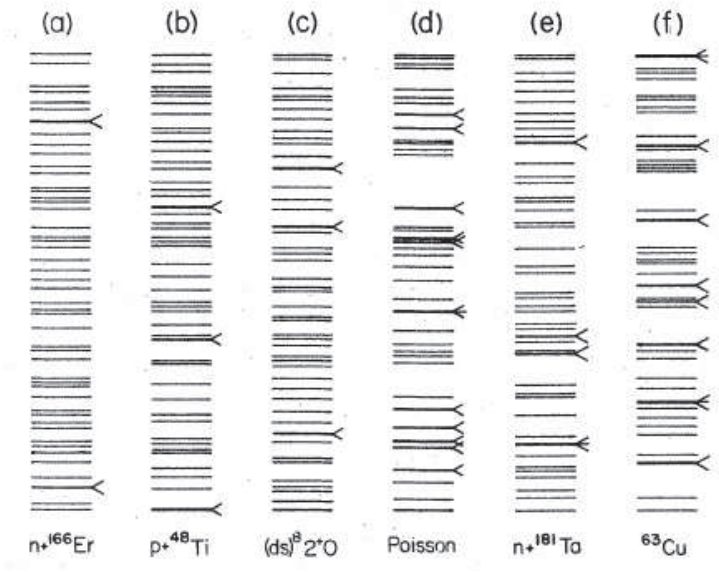
The average density: take a collection of subsets of energy levels and for each subset compute the number of levels per energy interval, $\rho(E)$.

Then average that density over the collection.

# Statistical analysis of spectra.

The idea is to focus on a large subset of energy levels as a *representative ensemble*.

For example, look at a subset of nuclear energy levels



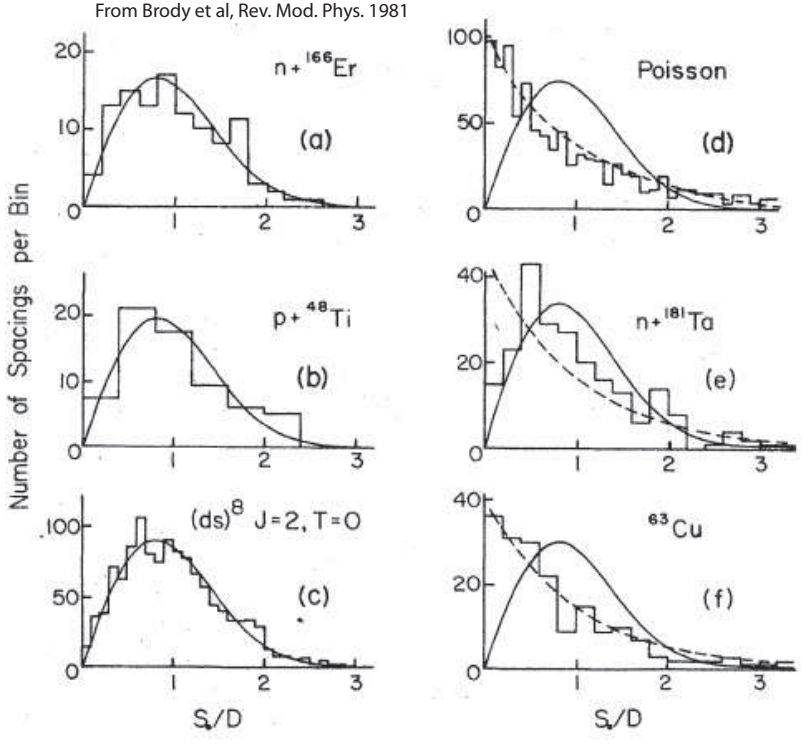

From Brody et al, Rev. Mod. Phys. 1981

These levels have the same symmetry and the energy scale has been chosen so that the average spacing is the same.

# Statistical analysis of spectra.

Look at a histogram of nearest neighbour spacing for each case.

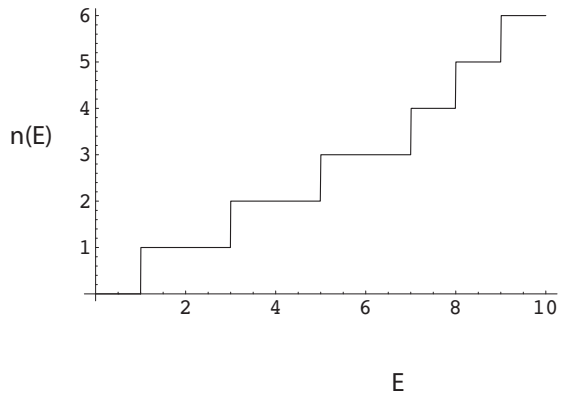
From Brody et al, Rev. Mod. Phys. 1981




Even though the spectra look different they have a *statistical equivalence*. The solid line is a theoretical model called the Wigner distribution.

# Statistical analysis of spectra.

The cumulative number of levels up to a large energy over some subset: $N(E)$




The energy density

$$\rho(E) = \frac{dN}{dE} = \sum_n \delta(E - E_n)$$

We can also take the average, $\overline{\rho(E)}$, of this over different subsets of energy levels.

# Random matrices.

We model the statistical fluctuations of a subset of real energy levels by looking at the eigenvalues of *random matrices* drawn from a suitable ensemble.

It must be Hermitian $H^\dagger = H$

In the eigenbasis every Hamiltonian is real, but in an arbitrary basis it is complex hermitian in general.

We can restrict all the matrices in a given ensemble to have a particular symmetry, eg real symmetric, to reflect some known symmetry in the real eigenvalue problem.

We chose the matrix elements as random variables according to some distribution, eg Gaussian.

# Random matrices.

If the Hamiltonian is invariant under time reversal and does not contain spin half interactions, it can always be chosen as real.

This property is preserved under orthogonal transformations.

$$H' = OHO^T \qquad OO^T = 1$$

In this case we must restrict the class of unitary transformations to be simply orthogonal matrices, and we will refer to the ensemble of random matrices with this property as an *orthogonal ensemble*.

If matrix elements are gaussian distributed, we have the *gaussian orthogonal ensemble* (GOE).

# Random matrices.

Gaussian *symplectic* ensemble (GSE). Systems with time reversal invariance and with spin interactions, eg spin-orbit interaction

$$H = g\vec{L} \cdot \vec{S}$$

where

$$\vec{L} = -i\hbar\vec{r} \times \vec{p}$$

is orbital angular momentum and $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ is the spin of the particle and $\vec{\sigma} = \sigma_x\vec{x} + \sigma_y\vec{y} + \sigma_z\vec{z}$ and $\sigma_\alpha$ are the Pauli matrices.

# Random matrices.

Such Hamiltonians are even dimensional and transform into each other under the *symplectic* transformation:

$$H' = SHS^R$$

where $S$ is a sympletic matrix which means it must satisfy $SS^R = 1$ with

$$S^R = ZS^T Z^{-1}$$

where

$$Z_{nm} = i\delta_{nm}\sigma_y$$

where

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

# Gaussian orthogonal ensemble.

A real symmetric matrix in $N$ dimensions requires $N(N+1)/2$ real numbers to specify the matrix elements.

# Gaussian orthogonal ensemble.

A real symmetric matrix in $N$ dimensions requires $N(N+1)/2$ real numbers to specify the matrix elements.

The joint probability distribution $p(H_{11}, H_{12}, \ldots, H_{NN})$ is then invariant under orthogonal transformations
$p(H_{11}, H_{12}, \ldots, H_{NN}) = p(H'_{11}, H'_{12}, \ldots, H'_{NN})$.

# Gaussian orthogonal ensemble.

A real symmetric matrix in $N$ dimensions requires $N(N+1)/2$ real numbers to specify the matrix elements.

The joint probability distribution $p(H_{11}, H_{12}, \ldots, H_{NN})$ is then invariant under orthogonal transformations $p(H_{11}, H_{12}, \ldots, H_{NN}) = p(H'_{11}, H'_{12}, \ldots, H'_{NN})$.

The trace of any matrix is invariant under orthogonal transformation $\mathrm{tr}(OAO^T) = \mathrm{tr}(A)$.

# Gaussian orthogonal ensemble.

A real symmetric matrix in $N$ dimensions requires $N(N+1)/2$ real numbers to specify the matrix elements.

The joint probability distribution $p(H_{11}, H_{12}, \ldots, H_{NN})$ is then invariant under orthogonal transformations $p(H_{11}, H_{12}, \ldots, H_{NN}) = p(H'_{11}, H'_{12}, \ldots, H'_{NN})$.

The trace of any matrix is invariant under orthogonal transformation $\mathrm{tr}(OAO^T) = \mathrm{tr}(A)$.

The joint distribution must be a function of the trace of powers of the Hamiltonian;

$$p(H_{11}, H_{12}, \ldots, H_{NN}) = f[\mathrm{tr}H, \mathrm{tr}(H^2), \ldots]$$

# Gaussian orthogonal ensemble.

If we now require the matrix elements to be uncorrelated

$$p(H_{11}, H_{12}, \ldots, H_{NN}) = p(H_{11})p(H_{12}) \ldots p(H_{NN})$$

then

$$p(H_{11}, H_{12}, \ldots, H_{NN}) = C \exp\left[-B\mathrm{tr}(H) - A\mathrm{tr}(H^2)\right]$$

As this is Gaussian in form, we can shift the average to ensure that $B = 0$. The normalisation fixes the factor $C$,

$$\int p(H_{11}, H_{12}, \ldots, H_{NN}) dH_{11} \ldots dH_{NN} = 1$$

We are thus led to the distribution

$$p(H_{11}, H_{12}, \ldots, H_{NN}) = \left(\frac{A}{\pi}\right)^{N/2} \left(\frac{2A}{\pi}\right)^{N(N-1)/2} \exp\left(-A \sum_{n,m} H_{nm}^2\right)$$

# Gaussian orthogonal ensemble.

The eigenvalue distribution is given by

$$P(E_1, E_2, \ldots, E_N) \sim \prod_{n>m} (E_n - E_m) \exp\left(-A \sum_n E_n^2\right)$$

# Gaussian orthogonal ensemble.

Calculate the nearest neighbour spacing distribution. We will only consider the $2 \times 2$ case as the results are independent of the dimension.

$$\begin{aligned} p(s) &= \int_{-\infty}^{\infty} dE_1 \int_{-\infty}^{\infty} dE_2 P(E_1, E_2) \delta(s - |E_1 - E_2|) \\ &= C \int_{-\infty}^{\infty} dE_1 \int_{-\infty}^{\infty} dE_2 |E_1 - E_2| \exp\left(-A \sum_n E_n^2\right) \\ &\quad \times \delta(s - |E_1 - E_2|) \end{aligned}$$

# Gaussian orthogonal ensemble.

the two constants are fixed by

$$\begin{aligned} \int_0^\infty p(s)ds &= 1 \\ \int_0^\infty sp(s)ds &= 1 \end{aligned}$$

Thus

$$p(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4}s^2\right)$$

# Gaussian unitary and symplectic ensemble.

For completeness we give the eigenvalue spacing distributions for the GUE and the GSE as well as the GOE.

$$p(s) = \begin{cases} \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right) & \text{(GOE)} \\ \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi} s^2\right) & \text{(GUE)} \\ \frac{2^{18}}{3^6 \pi^3} s^4 \exp\left(-\frac{\pi}{4} s^2\right) & \text{(GSE)} \end{cases} \tag{1}$$

Note the dependence at small spacing. The GOE is linear, the GUE is quadratic and the GSE is quartic.

In the case of Floquet operators we need to consider *Gaussian circular ensembles* as all eigenvalues lie on the unit circle. There is an equivalent breakdown in terms of orthogonal, unitary and sympletic transformations.

# Average density of states.

*density of states* defined as

$$\rho(E) = \sum_n \delta(E - E_n)$$

In many cases we find that,

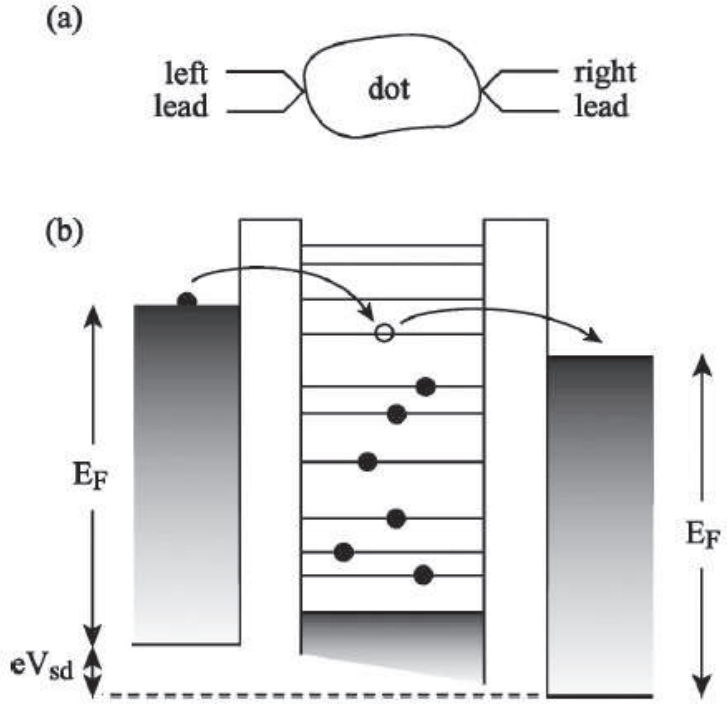
$$\rho(E) = \overline{\rho(E)} + \rho_{osc}(E)$$

The average here is a spectral average taken over the energy eigenvalues. This can often be replaced to a good approximation by an average over some ensemble of random matrices provided that the number of levels is very large.

This is a kind of ergodic theorem for energy eigenvalues and underpins the BGS conjecture.

# Back to quantum dots.

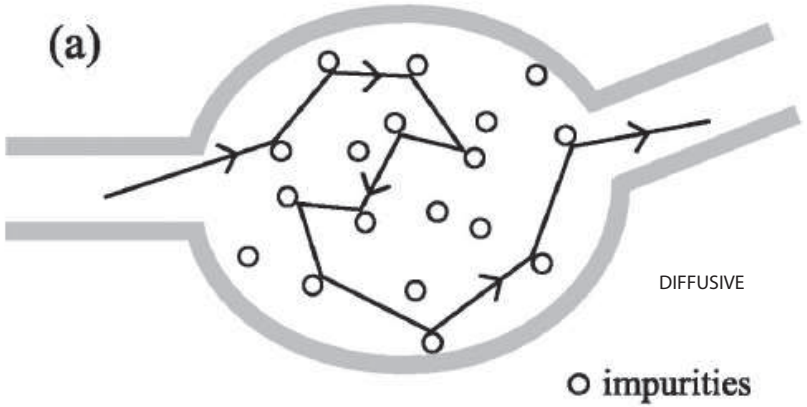
Look at tunneling through quasi-bound states



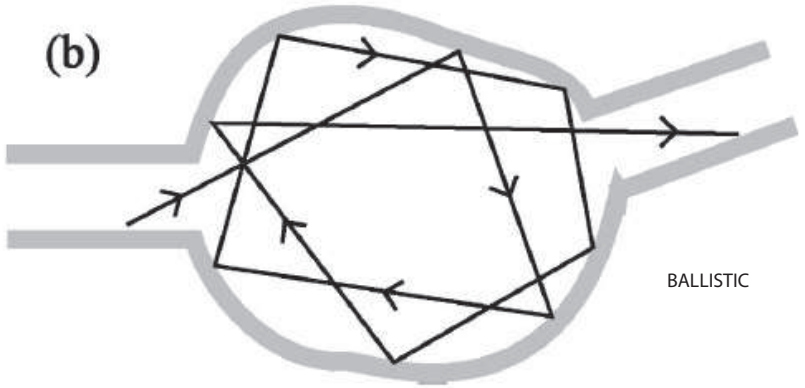

from Alhassid Rev. Mod Phys. fig 2

# Back to quantum dots.

Ballistic regime, many elastic reflections from the walls before exiting the dot.

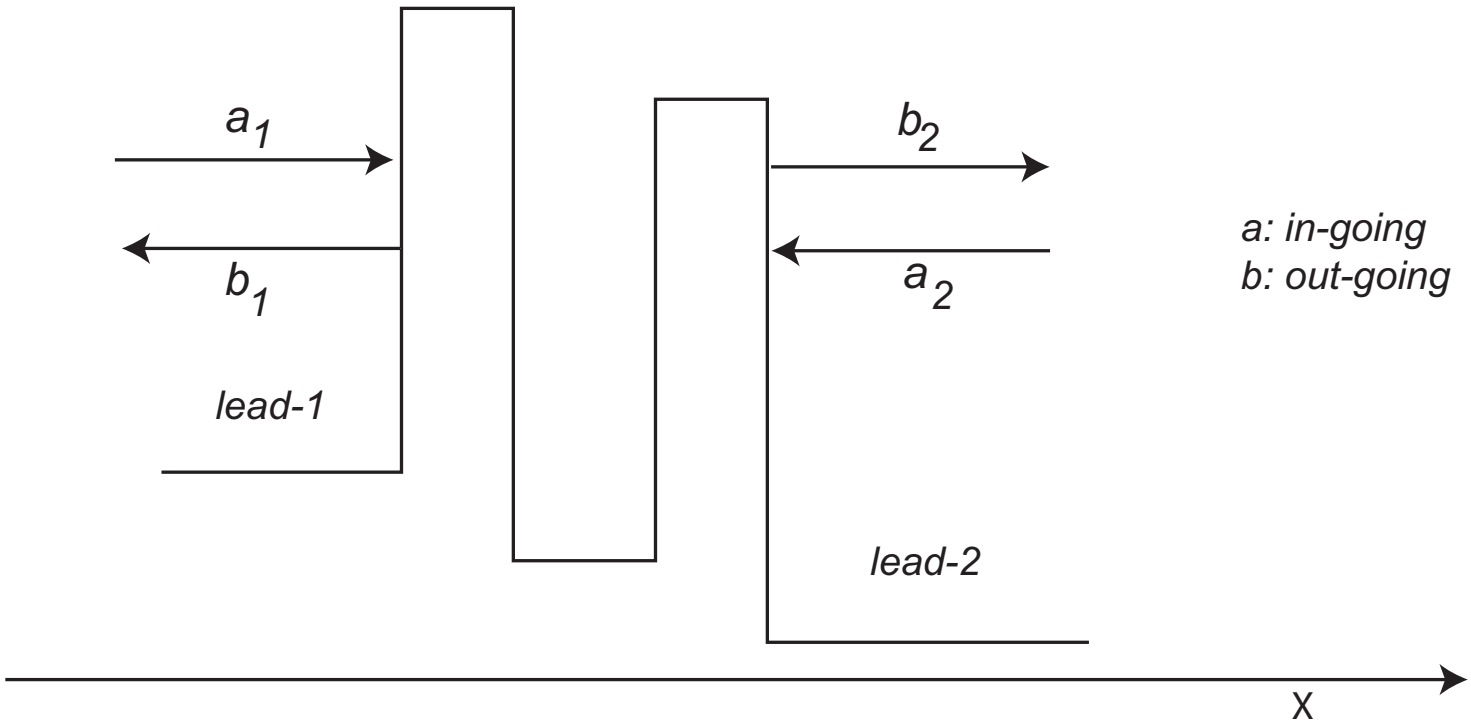



from Alhassid Rev. Mod Phys. fig 5



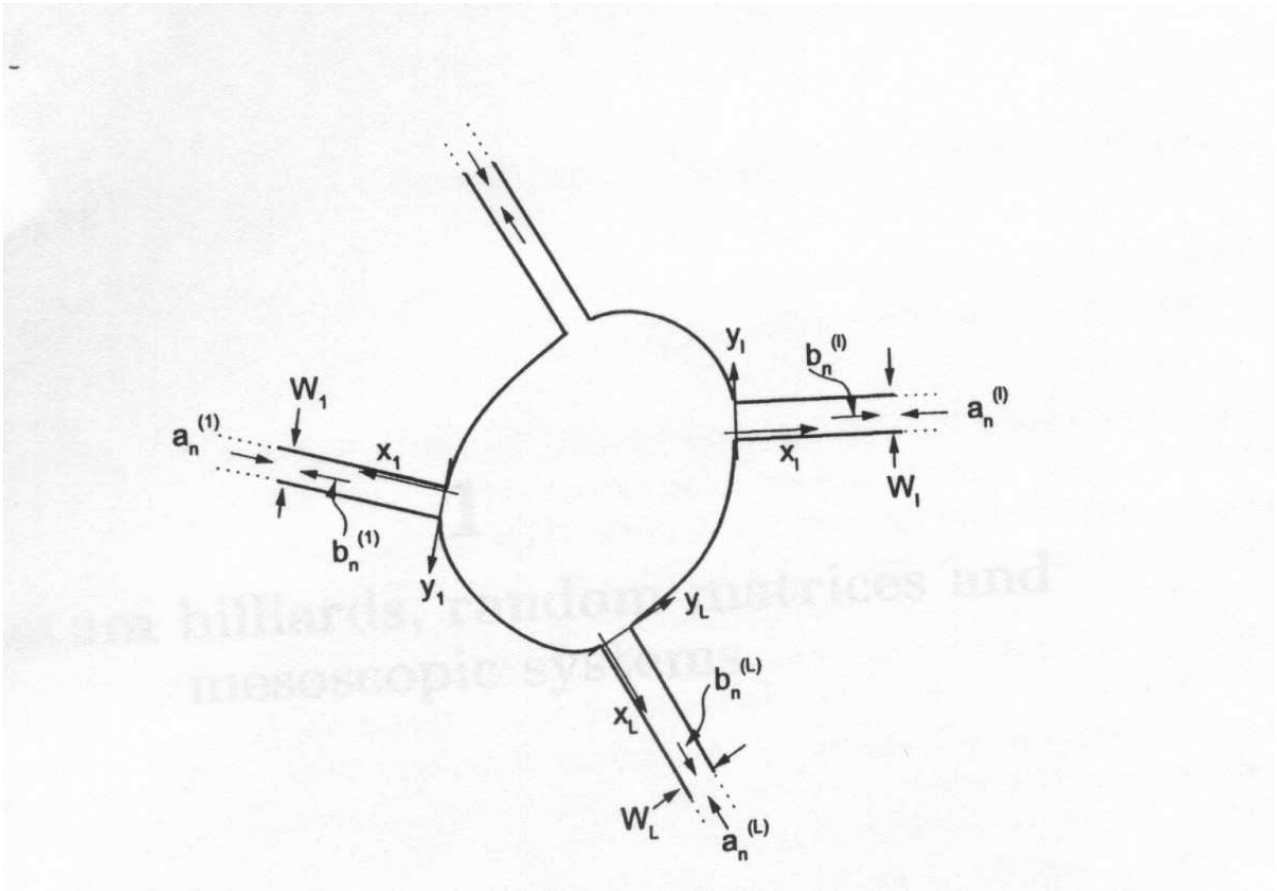
# Modes.

Define amplitudes for in-going or out-going states of definite momentum




$$b_1 = ra_1 + t'a_2$$
$$b_2 = ta_1 + r'a_2$$

# Multiple leads.




A 2d mesoscopic cavity connected to $L$ one dimensional leads or waveguides. The arrows indicate ingoing, $a_n^{(l)}$ or outgoing waves, $b_n^{(l)}$. In waveguide $l$ there can be $N_l$ such modes. (from Mello and Baranger, Waves Random Media, **9**, 105 (1999)).

# Conductance: Landauer-Buttiker.

Landauer and Buttiker: the conductance of a single channel of transmission probability $T$ is

$$G = \frac{2e^2}{h} T$$

where $T = |t_{ab}|^2$ and $t_{ab}$ is the relevant amplitude for scattering from lead a to lead b.
For the two lead case,

$$G = \frac{2e^2}{h} \mathrm{tr}(tt^\dagger)$$

Find the eigenvalues, $\tau_\alpha$, of $tt^\dagger$

$$G = \frac{2e^2}{h} \mathrm{tr} \sum_\alpha \tau_\alpha$$

**The conductance is determined by the eigenvalues of a hermitian matrix.**

# Conductance: Landauer-Buttiker.

Electrons are not neutral billiard balls but carry a charge. The energy required to put a single electron into the cavity is

$$E_c = \frac{e^2}{2C} \tag{2}$$

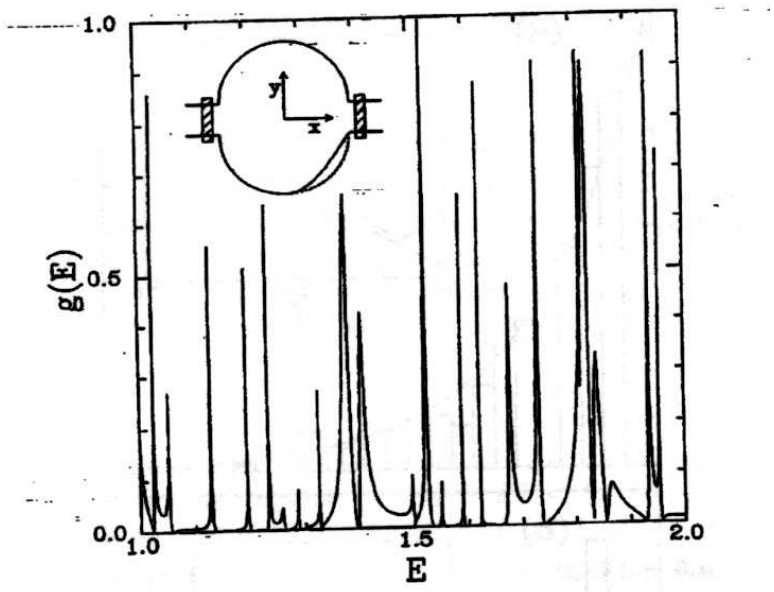
A quantum dot has a very small capacitance, C. Expect $E_c >> \Delta E = E_n - E_{n-1}$, the billiard spectral gaps.

A plot of conductance versus energy(= bias voltage) will show peaks spaced equally by the charging energy. This is called a *Coulomb blockade*.

The *height* of each peak will be determined by the scattering matrix.

# Experiments

Jalabert, Stone and Alhassid (Phys. Rev. Letts. **68**, 3468, (1992).) A stadium was desymmetrised by replacing one quarter circle by a cosine curve (insert in figure 6.7). The conductance peak heights, $g_n$ are, apparently random.




Find that a GOE of random matrices can model the mesoscopic conductance of a stadium.

# Experiments

A more detailed analysis based on random matrix theory (Alhassd and Lewenkopf, Phys. Rev. B **55**, 7749 (1997)), shows that the distribution of peak heights for a symmetric double lead chaotic mesoscopic cavity is

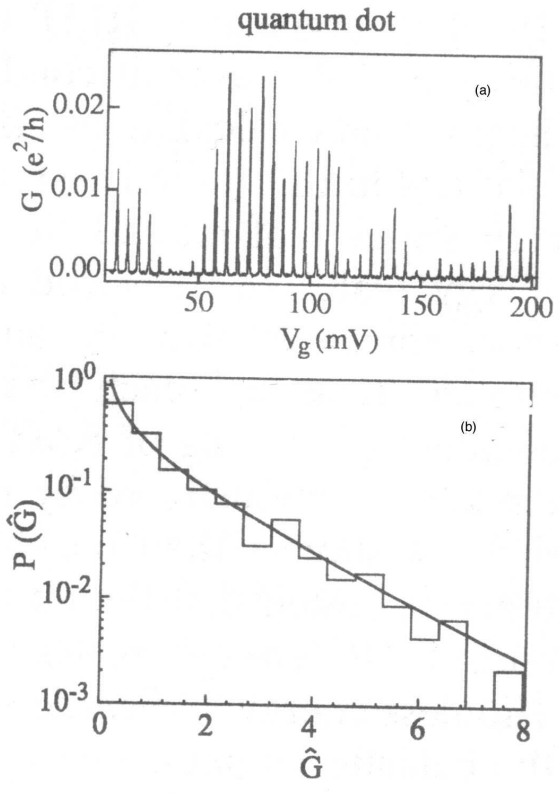
$$P(g) = \sqrt{\frac{1}{\pi s_0 g}} e^{-g/s_0}$$

where $s_0$ determines the average conductance peak height

$$\mathcal{E}(g) = \frac{s_0}{2}$$

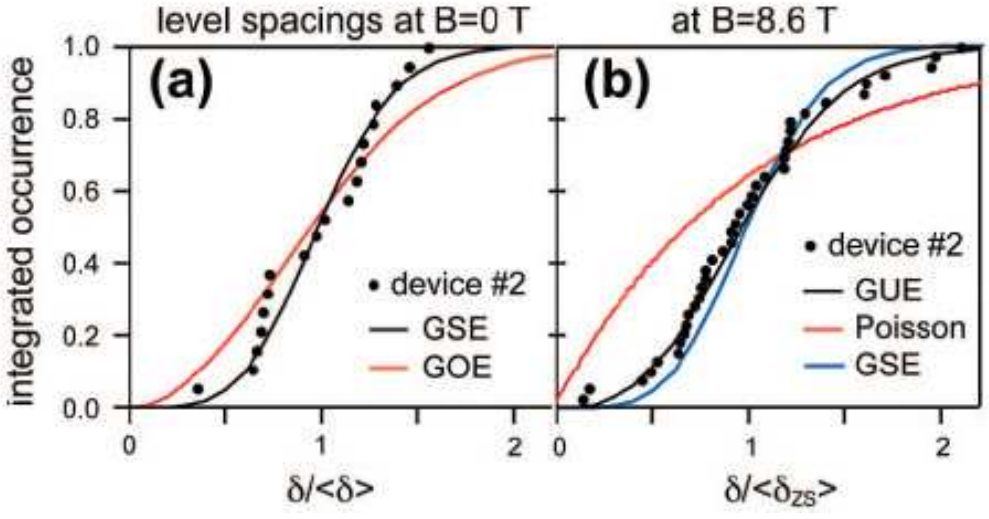
# Experiments

This theory is found to be in very good agreement with experiment of Folk et al. Phys. Rev. Lett. **76**, 1699 (1996).



# Experiments

*Measurement of Discrete Energy-Level Spectra in Individual Chemically Synthesized Gold Nanoparticles.*
Kuemmeth et al., NANO LETTERS (2008).




## Integrated probability distribution of level spacings.

For a chaotic quantum dot with strong spin-orbit coupling, RMT predicts that the level spacings for B=0 should be described by a Gaussian symplectic ensemble (spin rotation invariance is preserved) with a transition to a Gaussian unitary ensemble for large magnetic fields where time reversal symmetry and spin-rotation symmetry are broken