MATH4104: Quantum nonlinear dynamics.
Lecture Two.
Review of quantum theory.

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Two quantum principles.

THE QUANTUM PRINCIPLE I.

The physical universe is irreducibly random.

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The physical universe is irreducibly random.

Given *complete* knowledge of the state of a physical system, there is at least one measurement the results of which are completely random.

(Contrast classical mechanics: Given complete knowledge of the state of a physical system, the results of all measurements are completely certain.)

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CERTAINTY WITHIN UNCERTAINTY.

Probability amplitudes.

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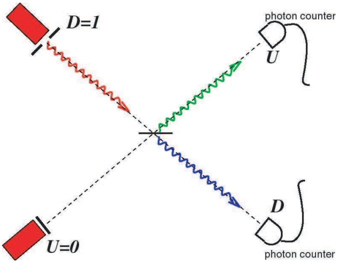
Given a list of probability amplitudes for one kind of measurement, we can generate the list for *all* physical measurements.

A simple experiment with light.

Photons at a beam splitter.

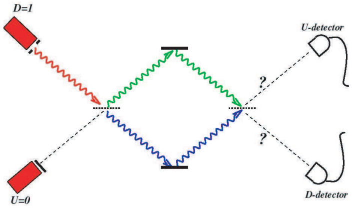
What happens if we use light of such low intensity that it has only one photon ?



The behaviour of any given photon is as random as a coin toss. The best we can do is give the odds (even in this case). This looks like a simple coin-toss.

A simple experiment with light.

'toss' the photon again:



After the first beam splitter we use two perfectly reflecting mirrors, one in the downward path and one in the upward path to direct the photon back onto another, identical, beam splitter. The photon counters are now moved back to monitor the beams after the second beam splitter.

What is the probability to detect a photon at, say, the upper detector, $P(U)$?

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There are four possible *histories* for a photon: it can be reflected at the first beam-splitter and the second beam-splitter (RR), or it can be transmitted at both, (TT), or it be reflected at the first and transmitted at the second, (RT) or vice-versa (TR).

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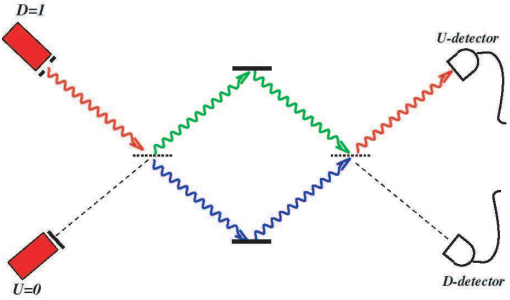
Each case is equally likely, so we assign a probability of 1/4 to each history, $P(RR) = P(TT) = P(RT) = P(TR)$. Detection at U can be achieved in two ways: RR and TT. Bayes' rule,

$$P(U) = P(RR) + P(TT) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

A simple experiment with light.

When the experiment is done, the results are completely different: the photon is *always* detected at U. Detection at U is certain, so that $P(U) = 1$.



In this experiment we see how quantum systems can exhibit both irreducible uncertainty and certainty depending on how we interrogate the system.

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The quantum state is then a list with just two (complex) elements, (t, r), such that $P(D) = |t|^2$ and $P(U) = |r|^2$.
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Refer to r as the probability amplitude for a photon to be reflected and t is the probability amplitude for the photon to be transmitted.

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Here a quantum state is an ordered pair of probability amplitudes: first entry represents the amplitude for a photon to be counted in the D-direction and the second entry corresponds to a photon being counted in the U-direction.

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Does, $(0, 1) \to (r, t)$?

There is a problem with this in the case of a 50/50 beam splitter, which highlights an important physical principle which we pause to consider.

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the probability of transmission and reflection, which are equal to t^2 and r^2 are the same for D and U input states.

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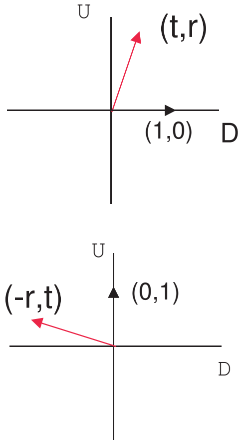

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A general state is then $|\psi\rangle = t|D\rangle + r|U\rangle$.

Quantum mechanics with particles.

A free particle is one that is not acted on by a force. In one dimension Newtonian physics gives the position as a function of time as

$$x(t) = x_0 + p_0 t/m$$

where x_0, p_0 are initial position and momentum.

The kinetic energy, depends only on momentum

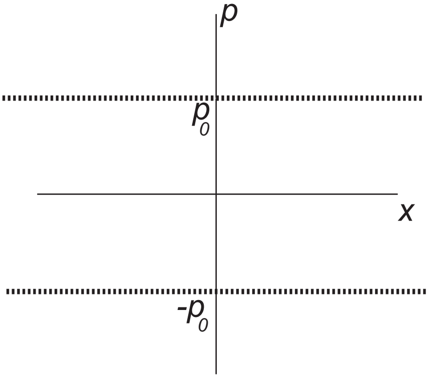
$$E = \frac{p_0^2}{2m}$$

is a constant of the motion

Note: $\pm p_0$ have the same energy.

Quantum mechanics with particles.

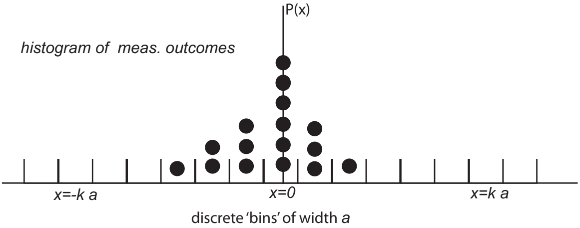
A phase-space picture for free particles of definite energy, E.
A *distribution* of states, all with the same energy,



Note, the position distribution is *uniform*.

Quantum mechanics with particles.

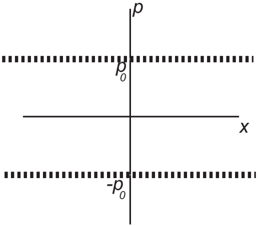
How to describe position measurements?

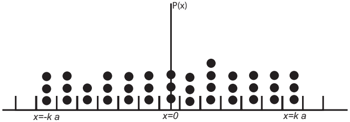


Plot a histograms of number of measurement results that lie in k-th bin.

Can make bin size — a — arbitrarily small.

Classical position distribution for states of definite energy.

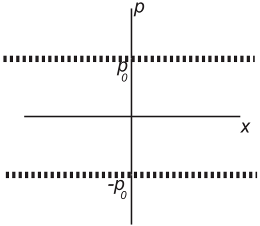




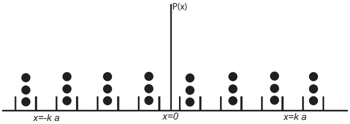
The distribution is uniform (within experimental sampling error).

Quantum position distribution for states of definite energy.

classical phase-space picture



quantum position distribution



The distribution is *oscillatory*.

There are some bins where the particle is *never* seen.

Quantum position distribution for states of definite energy.

Explanation: there are *two* indistinguishable ways to find a particle with definite energy at the $k-$th bin:

- in k and travelling with *positive* momentum and
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The last term is not always positive, so can cancel the first two terms... interference.

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$$\mathcal{A}(n:+p_0) = \frac{1}{\sqrt{N}} e^{-2\pi i n a p_0/h}$$

N is total number of bins and h is Planck's constant.

Note: ap_0 has units of action.

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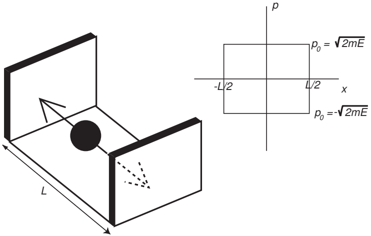
In continuum limit: $na \to x$

$$\mathcal{A}(x:+p_0) = \frac{1}{\sqrt{N}} e^{-ixp_0/\hbar}$$

$\hbar = h/2\pi$

Example: particle in a box.

An oscillator, but period depends on energy and motion is not harmonic.



The period,

$$T(E) = \frac{2mL}{p} = L\sqrt{\frac{2m}{E}}$$

The area is given by $2L\sqrt{2mE}$,

$$E_n = \frac{n^2 h^2}{8mL^2}$$

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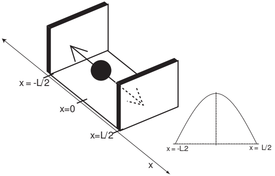

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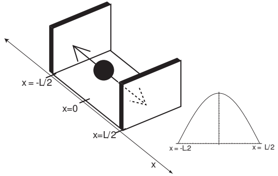

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$$P(x = \pm L/2) = 0$$

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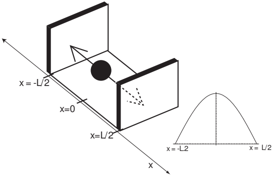

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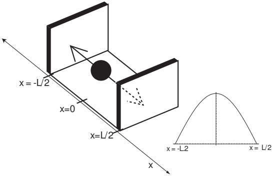

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The minimum allowed energy is $E_{min} = \frac{1}{2m}(p_0)^2_{min} = \frac{h^2}{8mL^2}$

Example: simple harmonic oscillator.

Simple harmonic oscillator: period is independent of energy.

$$x(t) = x_0 \cos(2\pi t/T)$$

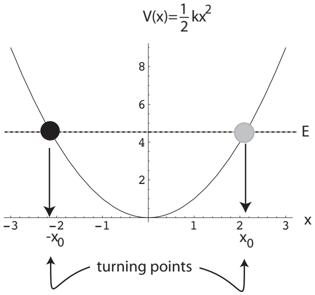
where T is the period of the motion.

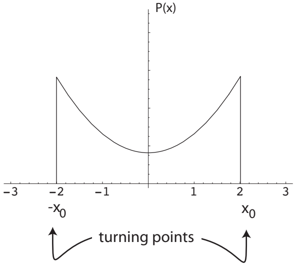
Surfaces of constant energy:

$$E = \frac{p^2}{2m} + \frac{k}{2}x^2 \tag{3}$$

Example: simple harmonic oscillator.

Classical position prob. distribution for a state of definite energy.





Example: simple harmonic oscillator.

Quantum SHO.
Fix energy, then

$$p(x) = \sqrt{2m\left(E - \frac{m\omega^2 x^2}{2}\right)}$$

The momentum is *not fixed*.

Schorödinger equation is required.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{m\omega^2}{2}\, x^2\psi(x) = E\psi(x)$$

In the form $\hat{H}\psi(x) = E\psi(x)$ where formally replace p in Hamiltonian by

$$p \rightarrow \hat{p} = -i\hbar\frac{d}{dx}$$

Example: simple harmonic oscillator.

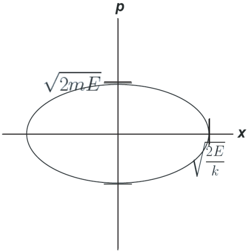
Solution to Schrödinger equation requires restriction on the energy,

$$E_n = \hbar\omega(n + \frac{1}{2}) \qquad n = 0, 1, 2 \ldots$$

the position measurement probability amplitudes are then

$$\psi_n(x) = (2\pi\Delta)^{-1/2} (2^n n!)^{-1/2} H_n\left(\frac{x}{2\sqrt{\Delta}}\right) e^{-\frac{x^2}{4\Delta}}$$

Quantisation of Sommerfeld-Epstein.



Area of the orbit is $E.T$

Sommerfeld's rule: only those orbits are allowed for which the action is an integer multiple of Planck's constant. The allowed energies are given by

$$E_n = nhf$$

Example: simple harmonic oscillator.

Check some averages

$$P_n(x) = |\psi_n(x)|^2$$

- $\int_{-\infty}^{\infty} dx\ P_n(x) = 1$
- $\int_{-\infty}^{\infty} dx\ xP_n(x) = 0$
- $\int_{-\infty}^{\infty} dx\ x^2 P_n(x) = \Delta(2n+1)$

Prove these results!