MATH4104: Quantum nonlinear dynamics. Lecture Four. Review of quantum theory: quantum dynamics.

G J Milburn

The University of Queensland

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Schrödinger dynamics.

Let $\psi_E(x)$ be the position probability amplitude for a state of a mechanical system with definite energy E prepared at the initial time $t = 0$

The state at later times $t > 0$ is then given by the *dynamical* Schrödinger rule:

$$\psi_E(x, t) = \psi_E(x)e^{-iEt/\hbar}$$

Note that the position probability density

$$P(x, t = 0) = P(x, t)$$

is stationary. States of definite energy are called *stationary states*

Schrödinger dynamics.

Let $\psi(x, t = 0)$ be an *arbitrary* quantum state of a mechanical system. How do we find the state at later times?

If we only measure the position of a particle, we have no information about its energy. Thus the probability amplitude to find a particle between x and $x + dx$ must be the sum of all the position probability amplitudes to find the particle at that point with definite energy E_i. By Feynman's rule:

$$\psi(x, t = 0) = \sum_E c(E)\psi_E(x)$$

The state at later times $t > 0$ are then give by

$$\psi(x, t = 0) = \sum_E c(E)\psi_E(x)e^{-iEt/\hbar}$$

Schrödinger dynamics.

Suppose we measure the energy instead of the position for the state $\psi(x)$. What is the probability to get the result E

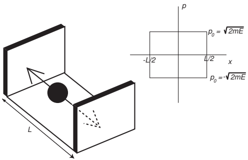
$$\psi(x, t = 0) = \sum_E c(E)\psi_E(x)$$

The probability amplitude to get the result E is just $c(E)$, so the probability is

$$P(E) = |C(E)|^2$$

Note that this does not change in time — a reflection of conservation of energy.

Schrödinger dynamics: square well.



The square well is a nonlinear oscillator because the period depends on the energy.

The allowed energies are

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2, \qquad n = 1, 2, 3, \ldots$$

and corresponding states of definite energy are

$$u_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Schrödinger dynamics: square well.

Consider the initial state:

$$\psi(x,0) = \sqrt{\frac{256}{126}}\sqrt{\frac{2}{L}}\left(\sin\left(\pi x/L\right)\right)^5$$

Exercise: verify that this state is normalised

We note that: $sin^5\theta = \frac{5}{8}\sin\theta - \frac{5}{16}\sin 3\theta + \frac{1}{16}\sin 5\theta$

Thus,

$$\psi(x,0) = \sqrt{\frac{256}{126}}\left[\frac{5}{8}u_1(x) - \frac{5}{16}u_3(x) + \frac{1}{16}u_5(x)\right]$$

Schrödinger dynamics: square well.

Apply Schrödinger's dynammical rule:

$$\psi(x,t) = \sqrt{\frac{256}{126}}\left[\frac{5}{8}u_1(x)e^{-i\omega_1 t} - \frac{5}{16}u_3(x)e^{-i\omega_3 t} + \frac{1}{16}u_5(x)e^{-i\omega_5 t}\right]$$

where $\omega_n = E_n/\hbar$.

Define a dimnesionless time: $\omega_1 t = \tau$

$$\psi(x,t) = \sqrt{\frac{256}{126}}\left[\frac{5}{8}u_1(x)e^{-i\tau} - \frac{5}{16}u_3(x)e^{-9i\tau} + \frac{1}{16}u_5(x)e^{-25i\tau}\right]$$

Prove that this state is periodic and find the period

Schrödinger dynamics: simple harmonic oscillator.

Consider the initial state

$$\psi(x, t = 0) = (2\pi\Delta)^{-1/4} e^{-(x-a)^2/4\Delta}$$

The initial mean and variance of position $\langle x \rangle = a, \ \mathrm{Var}(x) = \Delta$.
To find the state at time $t > 0$ we need to expand

$$\psi(x, t = 0) = \sum_{n=0}^{\infty} c_n u_n(x)$$

where

$$u_n(x) = (2\pi\Delta)^{-1/4} \left(2^n n!\right)^{-1/2} H_n\left(\frac{x}{\sqrt{2\Delta}}\right) e^{-\frac{x^2}{4\Delta}}$$

with corresponding energies

$$E_n = \hbar\omega(n + 1/2) \qquad n = 0, 1, \ldots$$

Schrödinger dynamics: simple harmonic oscillator.

Useful identity:

$$e^{2zy-z^2} = \sum_{n=0}^{\infty} H_n(y)\frac{z^n}{n!}$$

Thus

$$\begin{aligned}\psi(x, t=0) &= (2\pi\Delta)^{-1/4}\sum_{n=0}^{\infty}(2^n n!)^{-1/2} H_n\left(\frac{x}{\sqrt{2\Delta}}\right) e^{-x^2/4\Delta} \\ &\qquad \times (2^n n!)^{1/2}\left(\frac{a}{2\sqrt{2\Delta}}\right)^n \frac{1}{n!} e^{-a^2/8\Delta} \\ &= \sum_{n=0}^{\infty}\frac{(a/2\sqrt{\Delta})^n}{\sqrt{n!}} e^{-a^2/8\Delta} \quad u_n(x)\end{aligned}$$

where

$$c_n = \frac{(a/2\sqrt{\Delta})^n}{\sqrt{n!}} e^{-a^2/8\Delta}$$

Schrödinger dynamics: simple harmonic oscillator.

The probability to get result E_n if we measure energy on this state is

$$P_n = \frac{(a^2/4\Delta)^n}{n!} e^{-a^2/4\Delta} = \frac{\bar{n}}{n!} e^{-\bar{n}}$$

where

$$\bar{n} = \frac{a^2}{4\Delta}$$

Prove that the average energy is
$\langle E \rangle = \hbar\omega(\bar{n} + 1/2) = m\omega^2 a^2/2 + \hbar\omega/2$

Schrödinger dynamics: simple harmonic oscillator.

The initial state

$$\psi(x, t = 0) = (2\pi\Delta)^{-1/4} e^{-(x-a)^2/4\Delta}$$

thus evolves to

$$\psi(x, t) = e^{-i\omega t/2} \sum_{n=0}^{\infty} c_n u_n(x) e^{-i\omega n t}$$

$$c_n = \frac{(a/2\sqrt{\Delta})^n}{\sqrt{n!}} e^{-a^2/8\Delta}$$

1. **Prove that the state is periodic.**
2. **Prove that the average position as a function of time is** $\langle x(t) \rangle = a \cos \omega t$