MATH4104: Quantum nonlinear dynamics. Lecture Five.
Review of quantum theory: operators, mixed states and phase space.

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The simple harmonic oscillator.

Allowed energies

\[ E_n = \hbar \omega (n + 1/2) \quad n = 0, 1, \ldots \]

Energy eigenstates:

\[ u_n(x) = (2\pi\Delta)^{-1/4} (2^n n!)^{-1/2} H_n \left( \frac{x}{\sqrt{2\Delta}} \right) e^{-\frac{x^2}{4\Delta}} \]

Arbitrary state \( \psi(x) \):

\[ \psi(x) = \sum_{n=0}^{\infty} c_n u_n(x) \]

where

\[ c_n = \int_{-\infty}^{\infty} \, dx \, u_n^*(x) \psi(x) \]
The simple harmonic oscillator.

Alternative notation — a list of probability amplitudes for energy measurements

\[ \psi = (c_0, c_1, c_2, \ldots) = c_0(1, 0, 0, \ldots) + c_2(0, 1, 0, \ldots) + \ldots \]

Dirac notation

\[ |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle; \quad |\phi\rangle = \sum_{n=0}^{\infty} d_n |n\rangle \]

Define an *inner product* for two arbitrary states \(|\psi\rangle, |\phi\rangle\)

\[ \langle \phi | \psi \rangle = \sum_{n=0}^{\infty} d_n^* c_n \]

prove that this is also given by

\[ \langle \phi | \psi \rangle = \int_{-\infty}^{\infty} dx \; \phi^*(x) \psi(x) \]
The simple harmonic oscillator.

Average position

\[ \langle x \rangle = \int_{-\infty}^{\infty} dx \psi^*(x) \times \psi(x) \equiv \langle \psi | x | \psi \rangle \]

Average momentum?

\[ \langle p \rangle = \langle \psi | x | \psi \rangle = \int_{-\infty}^{\infty} dx \psi^*(x) \left( -i \hbar \frac{d}{dx} \right) \psi(x) \]

Recall, expansion over states of definite momentum,

\[ \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \ e^{-ipx/\hbar} \tilde{\psi}(p) \]

Substitute this and show that

\[ \langle p \rangle = \int_{-\infty}^{\infty} dpp \left| \tilde{\psi}(p) \right|^2 \]

Where we interpret \( P(p) = \left| \tilde{\psi}(p) \right|^2 \) as the momentum prob. density.
The simple harmonic oscillator.

Example: let the state be an energy eigenstate

\[ u_n(x) = (2\pi \Delta)^{-1/4} (2^n n!)^{-1/2} H_n \left( \frac{x}{\sqrt{2\Delta}} \right) e^{-\frac{x^2}{4\Delta}} \]

Find \( \langle p \rangle \), \( \langle p^2 \rangle \).

Use:

\[
\frac{d}{dx} H_n(x) = 2nH_{n-1}(x) \\
H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)
\]

to show

\[
\left( -i\hbar \frac{d}{dx} \right) u_n(x) = \frac{i\hbar}{\sqrt{4\Delta}} \left( (n + 1)^{1/2} u_{n+1}(x) - n^{1/2} u_{n-1}(x) \right)
\]

Prove this!
The simple harmonic oscillator.

Thus momentum averages in an energy eigenstate are:

$$\langle n|p|n \rangle = 0$$

$$\langle n|p^2|n \rangle = \frac{\hbar^2}{4\Delta}(2n + 1)$$

Prove these results!

Recall position averages, $$\langle n|x|n \rangle = 0$$ and $$\langle n|x^2|n \rangle = \Delta(2n + 1)$$
The simple harmonic oscillator.

Projection operators.

Recall that an arbitrary state in energy basis is

\[ |\psi\rangle = (c_0, c_1, c_2, \ldots) = c_0(1, 0, 0, \ldots) + c_2(0, 1, 0, \ldots) + \ldots \]

Project in the \(n\)’th basis direction:

\[ \psi \rightarrow (0, 0, \ldots, 0, c_n, 0, \ldots) = c_n|n\rangle \]

We can write this as

\[ \psi \rightarrow |n\rangle\langle n|\psi\rangle \]

Define the projection operator

\[ \Pi_n = |n\rangle\langle n| \]
The simple harmonic oscillator.

Measurement of a projection operator.

Given the state $|\psi\rangle$, what is the average value of $\Pi_n$?

\[
\langle \psi | \Pi_n | \psi \rangle = \langle \psi | n \rangle \langle n | \psi \rangle = |\langle n | \psi \rangle|^2 = |c_n|^2
\]

The average of the projection operator is a probability.
Define

\[ \hat{H} = \sum_{n=0}^{\infty} E_n \Pi_n \]

The average energy for state \( |\psi\rangle \) is given by

\[ \langle E \rangle = \langle \psi | \hat{H} | \psi \rangle \]

* This is called the spectral decomposition
Raising and lowering operators.

Define two new operators $a, a^\dagger$ by their action on energy eigenstates

$$a|n\rangle = n^{1/2}|n-1\rangle$$

$$a^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle$$

with $a|0\rangle = 0$.

**Prove that:**

$$\hat{H} = \hbar\omega (a^\dagger a + 1/2)$$

$$\hat{q} = \sqrt{\Delta} (a + a^\dagger)$$

$$\hat{p} = -i\frac{\hbar}{2\sqrt{\Delta}} (a - a^\dagger)$$

are the operators for energy, position and momentum respectively.
Let $\psi(x)$ be an arbitrary state in position representation.

$$\hat{q}\hat{p}|\psi\rangle \rightarrow x \left(-i\hbar \frac{d}{dx}\right) \psi(x)$$

$$\hat{p}\hat{q}|\psi\rangle \rightarrow \left(-i\hbar \frac{d}{dx}\right) x \psi(x)$$

$$= -i\hbar \psi(x) + x \left(-i\hbar \frac{d}{dx}\right) \psi(x)$$

Thus

$$(\hat{q}\hat{p} - \hat{p}\hat{x}) \psi = i\hbar |\psi\rangle$$

Define the \textit{canonical commutation relation}

$$[\hat{q}, \hat{p}] = (\hat{q}\hat{p} - \hat{p}\hat{x}) = i\hbar$$

Now show that

$$[a, a^\dagger] = 1.$$
Canonical commutation relations.

We can show that the state $|\psi_0\rangle$ with position prob. amp.

$$\psi_0(x) = \langle x|\psi_0\rangle \propto \exp(-x^2/4\Delta)$$

is an eigenstate of $a$ with eigenvalue 0.

**Show this using the position representation of $\hat{p}$ as $-i\hbar \frac{\partial}{\partial x}$**.

It is eigenstate of the Hamiltonian with eigenvalue $\hbar \omega/2$, the lowest eigenvalue of the Hamiltonian.

The ground state of the SHO is a minimum uncertainty state with $\langle \hat{q} \rangle = \langle \hat{p} \rangle = 0$ and a characteristic length given by

$$\sqrt{\Delta} = \sqrt{\hbar/2m\omega}$$
Oscillator coherent states.

This state is defined as an eigenstate of the annihilation operator

\[ a|\alpha\rangle = \alpha|\alpha\rangle \quad (1) \]

where \( \alpha \) is a complex number (because \( \hat{a} \) is not an Hermitian operator).

There are no such eigenstates of the creation operator \( a^\dagger \).

**Show this.** Assume that there exists states \( |\beta\rangle \) such that \( a^\dagger |\beta\rangle = \beta |\beta\rangle \) and consider the inner product \( \langle n|(a^\dagger)^{n+1}|\beta\rangle \). Hence show that the inner product of \( |\beta\rangle \) with any number state is zero.
Oscillator coherent states.

Expansion in energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle.$$
Oscillator coherent states.

Expansion in energy eigenstates:

\[ |\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle. \]

Since \( a |\alpha\rangle = \alpha |\alpha\rangle \) we get

\[ \sum_{n=0}^{\infty} \sqrt{n} c_n |n - 1\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle. \]
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Equating the coefficients

$$c_{n+1} = \frac{\alpha}{\sqrt{n + 1}} c_n.$$
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so that $c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$. Choosing $c_0$ real and normalizing the state,

$$|\alpha\rangle = \exp \left( -|\alpha|^2 / 2 \right) \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$
Oscillator coherent states.

Average energy

$$\hbar \omega \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = \hbar \omega \alpha^* \langle \alpha | \alpha \rangle \alpha = \hbar \omega |\alpha|^2.$$
Oscillator coherent states.

Average energy

$$\hbar \omega \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = \hbar \omega \alpha^* \langle \alpha | \alpha \rangle \alpha = \hbar \omega |\alpha|^2.$$  

The energy probability distribution for a coherent state is

$$P_n = |\langle n | \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{(|\alpha|^2)^n}{n!}.$$
Oscillator coherent states.

Average energy

\[ \hbar \omega \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = \hbar \omega \alpha^* \langle \alpha | \alpha \rangle \alpha = \hbar \omega |\alpha|^2. \]

The energy probability distribution for a coherent state is

\[ P_n = |\langle n | \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{(|\alpha|^2)^n}{n!} \]

a Poisson distribution, with variance equal to the mean,

\[ \langle (a^\dagger a)^2 \rangle - \langle a^\dagger a \rangle^2 = |\alpha|^2 \]

Verify this, either from the distribution, \( P_n \), or directly from the coherent state using the commutation relations for \( a \) and \( a^\dagger \).
Oscillator coherent states.

Now show that,

\[ \langle \alpha|\hat{q}|\alpha \rangle = 2\sqrt{\Delta} \text{ Re}[\alpha] \]
\[ \langle \alpha|\hat{p}|\alpha \rangle = \sqrt{\hbar/\Delta} \text{ Im}[\alpha] \]
\[ \langle \alpha|(\Delta \hat{q})^2|\alpha \rangle = \Delta \]
\[ \langle \alpha|(\Delta \hat{p})^2|\alpha \rangle = \hbar^2/4\Delta \]
\[ \langle \alpha|\Delta \hat{q}\Delta \hat{p} + \Delta \hat{p}\Delta \hat{q}|\alpha \rangle = 0 \]

That is, a coherent state is a minimum uncertainty state.

A large value for \( \alpha \) suggests a \textit{semiclassical state}. 
Oscillator coherent states.

Because \( a \) is not an Hermitian operator, the coherent states are not orthogonal.

\[
|\langle \alpha | \alpha' \rangle|^2 = \exp(-|\alpha - \alpha'|^2).
\]

If \( \alpha \) and \( \alpha' \) are very different (as they would be if they represent two macroscopically distinct states) then the two coherent states are very nearly orthogonal.

A useful identity

\[
\int d^2 \alpha |\alpha\rangle \langle \alpha| = \pi \hat{1}.
\]

Show this using the expansion in terms \( |n\rangle \). The result \( n! = \int_0^\infty dxe^{-x}x^n \) may be useful.
Oscillator coherent states.

Dynamics.

Use the expansion of $|\alpha\rangle$ over the energy eigenstates to show that an initial coherent state evolves in time as

$$|\alpha\rangle \rightarrow |\alpha e^{-i\omega t}\rangle$$
The Husimi function

Given an arbitrary state of a mechanical system, $|\psi\rangle$, define

$$Q(\alpha, \alpha) = |\langle \alpha | \psi \rangle|^2$$

This is the average value of the projection operator

$$\Pi(\alpha) = |\alpha\rangle\langle \alpha|$$

As

$$\int d^2 \alpha |\alpha\rangle\langle \alpha| = \pi \cdot \hat{1}$$

we have that

$$\frac{1}{\pi} \int d^2 \alpha Q(\alpha, \alpha) = 1$$

As $Q(\alpha, \alpha)$ is clearly positive and normalised, it has an interpretation as a probability density on phase space, $q, p$, where

$$q = 2\sqrt{\Delta} \text{ Re}[\alpha], \quad p = \sqrt{\hbar/\Delta} \text{ Im}[\alpha]$$
The Husimi function

Examples:

SHO energy eigenstate, $|n\rangle$

$$Q_n(\alpha, \alpha) = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$
The Husimi function

Examples:

A coherent state, $|\alpha_0\rangle$

$$Q_{\alpha_0}(\alpha, \alpha) = e^{-|\alpha - \alpha_0|^2}$$