MATH4104: Quantum nonlinear dynamics. Lecture Five. Review of quantum theory: operators, mixed states and phase space.

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The simple harmonic oscillator.

Allowed energies

$$E_n = \hbar\omega(n + 1/2) \qquad n = 0, 1, \ldots$$

Energy *eigenstates*:

$$u_n(x) = (2\pi\Delta)^{-1/4} (2^n n!)^{-1/2} H_n\left(\frac{x}{\sqrt{2\Delta}}\right) e^{-\frac{x^2}{4\Delta}}$$

Arbitrary state $\psi(x)$:

$$\psi(x) = \sum_{n=0}^{\infty} c_n u_n(x)$$

where

$$c_n = \int_{-\infty}^{\infty} dx \; u_n^*(x)\psi(x)$$

The simple harmonic oscillator.

Alternative notation — a list of probability amplitudes for energy measurements

$$\psi = (c_0, c_1, c_2, \ldots) = c_0(1, 0, 0, \ldots) + c_2(0, 1, 0, \ldots) + \ldots$$

Dirac notation

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle; \qquad |\phi\rangle = \sum_{n=0}^{\infty} d_n |n\rangle$$

Define an *inner product* for two arbitrary states $|\psi\rangle, \ |\phi\rangle$

$$\langle\phi|\psi\rangle = \sum_{n=0}^{\infty} d_n^* c_n$$

prove that this is also given by

$$\langle\phi|\psi\rangle = \int_{-\infty}^{\infty} dx \ \phi^*(x)\psi(x)$$

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Average position

$$\langle x\rangle = \int_{-\infty}^{\infty} dx\psi^*(x)\ x\ \psi(x) \equiv \langle\psi|x|\psi\rangle$$

Average momentum?

$$\langle p\rangle = \langle\psi|x|\psi\rangle = \int_{-\infty}^{\infty} dx\psi^*(x)\ \left(-i\hbar\frac{d}{dx}\right)\ \psi(x)$$

Recall, expansion over states of definite momentum,

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\ dx\ e^{-ipx/\hbar}\tilde{\psi}(p)$$

Substitute this and show that

$$\langle p\rangle = \int_{-\infty}^{\infty}\ dpp|\tilde{\psi}(p)|^2$$

Where we interpret $P(p) = |\tilde{\psi}(p)|^2$ as the momentum prob. density.

The simple harmonic oscillator.

Example: let the state be an energy eigenstate

$$u_n(x) = (2\pi\Delta)^{-1/4}\,(2^n n!)^{-1/2}\,H_n\left(\frac{x}{\sqrt{2\Delta}}\right)\,e^{-\frac{x^2}{4\Delta}}$$

Find $\langle p \rangle, \quad \langle p^2 \rangle$.

Use:

$$\begin{aligned} \frac{d}{dx}H_n(x) &= 2nH_{n-1}(x) \\ H_{n+1}(x) &= 2xH_n(x) - 2nH_{n-1}(x) \end{aligned}$$

to show

$$\left(-i\hbar\frac{d}{dx}\right)u_n(x) = \frac{i\hbar}{\sqrt{4\Delta}}\left[(n+1)^{1/2}u_{n+1}(x) - n^{1/2}u_{n-1}(x)\right]$$

Prove this!

The simple harmonic oscillator.

Thus momentum averages in an energy eigenstate are:

$$\begin{aligned}\langle n|p|n\rangle &= 0\\ \langle n|p^2|n\rangle &= \frac{\hbar^2}{4\Delta}(2n+1)\end{aligned}$$

Prove these results!

Recall position averages, $\langle n|x|n\rangle = 0$ and $\langle n|x^2|n\rangle = \Delta(2n+1)$

The simple harmonic oscillator.

Projection operators.

Recall that an arbitrary state in energy basis is

$$|\psi\rangle = (c_0, c_1, c_2, \dots) = c_0(1,0,0,\dots) + c_2(0,1,0,\dots) + \dots$$

Project in the n'th basis direction:

$$\psi \to (0,0,\dots,0,c_n,0,\dots) = c_n|n\rangle$$

We can write this as

$$\psi \to |n\rangle\langle n|\psi\rangle$$

Define the projection operator

$$\Pi_n = |n\rangle\langle n|$$

The simple harmonic oscillator.

Measurement of a projection operator.

Given the state $|\psi\rangle$, what is the average value of Π_n?

$$\begin{aligned} \langle\psi|\Pi_n|\psi\rangle &= \langle\psi|n\rangle\langle n|\psi\rangle \\ &= |\langle n|\psi\rangle|^2 \\ &= |c_n|^2 \end{aligned}$$

The average of the projection operator is a *probability*.

Operator representation*

Define

$$\hat{H} = \sum_{n=0}^{\infty} E_n \Pi_n$$

The the average energy for state $|\psi\rangle$ is given by

$$\langle E \rangle = \langle \psi | \hat{H} | \psi \rangle$$

* This is called the spectral decomposition

Raising and lowering operators.

Raising and lowering operators.

Define two new operators $a, a^\dagger$ by their action on energy eigenstates

$$\begin{aligned} a|n\rangle &= n^{1/2}|n-1\rangle \\ a^\dagger|n\rangle &= (n+1)^{1/2}|n+1\rangle \end{aligned}$$

with $a|0\rangle = 0$.

Prove that:

$$\begin{aligned} \hat{H} &= \hbar\omega(a^\dagger a + 1/2) \\ \hat{q} &= \sqrt{\Delta}(a + a^\dagger) \\ \hat{p} &= -i\frac{\hbar}{2\sqrt{\Delta}}(a - a^\dagger) \end{aligned}$$

are the operators for energy, position and momentum respectively.

Canonical commutation relations..

Let $\psi(x)$ be an arbitrary state in position representation.

$$\hat{q}\hat{p}|\psi\rangle \to x\left(-i\hbar\frac{d}{dx}\right)\psi(x)$$

$$\begin{aligned}\hat{p}\hat{q}|\psi\rangle &\to \left(-i\hbar\frac{d}{dx}\right)x\psi(x)\\ &= -i\hbar\psi(x) + x\left(-i\hbar\frac{d}{dx}\right)\psi(x)\end{aligned}$$

Thus

$$(\hat{q}\hat{p} - \hat{p}\hat{x})\,\psi = i\hbar|\psi\rangle$$

Define the *canonical commutation relation*

$$[\hat{q},\hat{p}] = (\hat{q}\hat{p} - \hat{p}\hat{x}) = i\hbar$$

Now show that

$$[a, a^{\dagger}] = 1.$$

Canonical commutation relations.

We can show that the state $|\psi_0\rangle$ with position prob. amp.

$$\psi_0(x) = \langle x|\psi_0\rangle \propto \exp(-x^2/4\Delta)$$

is an eigenstate of a with eigenvalue 0.

Show this using the position representation of $\hat{p}$ as $-i\hbar\frac{\partial}{\partial x}$.

It is eigenstate of the Hamiltonian with eigenvalue $\hbar\omega/2$, the lowest eigenvalue of the Hamiltonian.

The ground state of the SHO is a minimum uncertainty state with $\langle\hat{q}\rangle = \langle\hat{p}\rangle = 0$ and a characteristic length given by

$$\sqrt{\Delta} = \sqrt{\hbar/2m\omega}$$

Oscillator coherent states.

This state is defined as an eigenstate of the annihilation operator

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{1}$$

where α is a complex number (because $\hat{a}$ is not an Hermitian operator).

There are no such eigenstates of the creation operator $a^\dagger$.

Show this. Assume that there exists states $|\beta\rangle$ such that $a^\dagger|\beta\rangle = \beta|\beta\rangle$ and consider the inner product $\langle n|(a^\dagger)^{n+1}|\beta\rangle$. Hence show that the inner product of $|\beta\rangle$ with any number state is zero.

Oscillator coherent states.

Expansion in energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle.$$

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Since $a|\alpha\rangle = \alpha|\alpha\rangle$ we get

$$\sum_{n=0}^{\infty} \sqrt{n} c_n |n-1\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle.$$

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Equating the coefficients

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}} c_n.$$

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so that $c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$. Choosing c_0 real and normalizing the state,

$$|\alpha\rangle = \exp\left(-|\alpha|^2/2\right) \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

Oscillator coherent states.

Average energy

$$\hbar\omega\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \hbar\omega\alpha^*\langle\alpha|\alpha\rangle\alpha = \hbar\omega|\alpha|^2.$$

Oscillator coherent states.

Average energy

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The energy probability distribution for a coherent state is

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{\left(|\alpha|^2\right)^n}{n!}$$

Oscillator coherent states.

Average energy

$$\hbar\omega\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \hbar\omega\alpha^*\langle\alpha|\alpha\rangle\alpha = \hbar\omega|\alpha|^2.$$

The energy probability distribution for a coherent state is

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{\left(|\alpha|^2\right)^n}{n!}$$

a Poisson distribution, with variance equal to the mean,

$$\left\langle (a^\dagger a)^2 \right\rangle - \left\langle a^\dagger a \right\rangle^2 = |\alpha|^2$$

Verify this, either from the distribution, P_n, or directly from the coherent state using the commutation relations for a and $a^\dagger$.

Oscillator coherent states.

Now show that,

$$
\begin{aligned}
\langle\alpha|\hat{q}|\alpha\rangle &= 2\sqrt{\Delta}\ \mathrm{Re}[\alpha] \\
\langle\alpha|\hat{p}|\alpha\rangle &= \sqrt{\hbar/\Delta}\ \mathrm{Im}[\alpha] \\
\langle\alpha|(\Delta\hat{q})^2|\alpha\rangle &= \Delta \\
\langle\alpha|(\Delta\hat{p})^2|\alpha\rangle &= \hbar^2/4\Delta \\
\langle\alpha|\Delta\hat{q}\Delta\hat{p}+\Delta\hat{p}\Delta\hat{q}|\alpha\rangle &= 0
\end{aligned}
$$

That is, a coherent state is a minimum uncertainty state.

A large value for α suggests a *semiclassical state*.

Oscillator coherent states.

Because a is not an Hermitian operator, the coherent states are not orthogonal.

$$|\langle\alpha|\alpha'\rangle|^2 = \exp(-|\alpha - \alpha'|^2).$$

If α and α' are very different (as they would be if they represent two macroscopically distinct states) then the two coherent states are very nearly orthogonal.

A useful identity

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = \pi\hat{1}.$$

Show this using the expansion in terms $|n\rangle$. The result $n! = \int_0^\infty dx x^n e^{-x}$ may be useful.

Oscillator coherent states.

Dynamics.

Use the expansion of $|\alpha\rangle$ over the energy eigenstates to show that an initial coherent state evolves in time as

$$|\alpha\rangle \to |\alpha e^{-i\omega t}\rangle$$

The Husimi function

Given an arbitrary state of a mechanical system, $|\psi\rangle$, define

$$Q(\alpha, \alpha) = |\langle\alpha|\psi\rangle|^2$$

This is the average value of the projection operator

$$\Pi(\alpha) = |\alpha\rangle\langle\alpha|$$

As

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = \pi \cdot \hat{1}$$

we have that

$$\frac{1}{\pi}\int d^2\alpha Q(\alpha, \alpha) = 1$$

As $Q(\alpha, \alpha)$ is clearly positive and normalised, it has an interpretation as a *probability density on phase space*, q, p, where

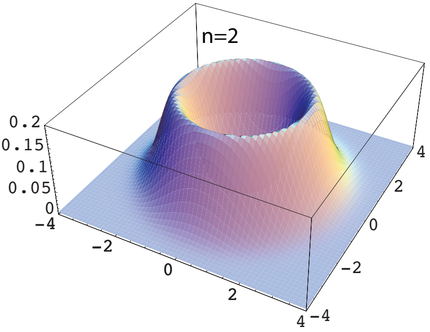
$$q = 2\sqrt{\Delta}\ \mathrm{Re}[\alpha], \qquad p = \sqrt{\hbar/\Delta}\ \mathrm{Im}[\alpha]$$

The Husimi function

Examples:

SHO energy eigenstate, $|n\rangle$

$$Q_n(\alpha, \alpha) = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$



The Husimi function

Examples:

A coherent state , $|\alpha_0\rangle$

$$Q_{\alpha_0}(\alpha,\alpha) = e^{-|\alpha-\alpha_0|^2}$$