MATH4104: Quantum nonlinear dynamics. Lecture Eight. Quantum dynamics in a periodic driven systems

G J Milburn

The University of Queensland

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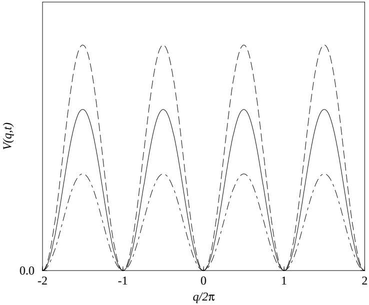
Particle in a periodic potential.

Include a modulation of the potential depth.

$$H(t) = \frac{p^2}{2} - \kappa(1 - 2\epsilon \cos t) \cos(q)$$



Particle in a periodic potential.

Stroboscopic (Floquet) dynamics.

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Only determine $(q(t), p(t))$ at integer multiples of the driving period, 2π.

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The classical stroboscopic dynamics of the variables q and p are determined by the recursive formula, *Floquet map*

$$(q', p') = F\big((q, p)\big) = \big(\bar{q}(q, p, 2\pi), \bar{p}(q, p, 2\pi)\big) \ ,$$

where $\bar{q}(q, p, t)$ and $\bar{p}(q, p, t)$ are determined by Hamilton's equations and the initial conditions $\bar{q}(q, p, 0) = q$ and $\bar{p}(q, p, 0) = p$.

Driven SHO.

Simple example: periodically driven SHO.

$$H = \frac{p_x^2}{2m} + \frac{m\omega^2}{2}x^2 + f_0 x \cos \Omega t$$

Hamilton's equations

$$\begin{aligned} \dot{x} &= p/m \\ \dot{p_x} &= -m\omega^2 x - f_0 \cos \Omega t \end{aligned}$$

Thus

$$\ddot{x} = -\omega^2 x - (f_0/m) \cos \Omega t$$

Solution:

$$x(t) = x(0) \cos \omega t + \frac{p_0}{m\omega} \sin \omega t - \frac{f_0}{m\omega} Imag \left[\int_0^t dt' e^{-i\omega(t-t')} \cos \Omega t' \right]$$

Driven SHO.

Quantum driven SHO

$$H = \hbar\omega a^\dagger a + \epsilon_0 (a + a^\dagger) \cos(\Omega t)$$

$$\epsilon_0 = f_0 \sqrt{\Delta}$$

with $\Delta = \hbar/(2m\omega)$
Equations of motion

$$\frac{da}{dt} = -i\omega a - i\epsilon_0 \cos(\Omega t)$$

Driven SHO.

Rotating frame (interaction picture)

$$\tilde{a}(t) = a(t)e^{i\Omega t}$$

$$\begin{aligned} \frac{d\tilde{a}}{dt} &= -i\delta\tilde{a} - i\epsilon_0 \frac{\left(e^{i\Omega t} + e^{-i\Omega t}\right)}{2} e^{i\Omega t} \\ &\approx -i\delta\tilde{a} - i\epsilon_0/2 \end{aligned}$$

for times $\Omega t >> 1$, the rotating wave approximation, and $\delta = \omega - \Omega$.

Driven SHO.

Solution

$$\tilde{a}(t) = a(0)e^{-i\delta t} - \frac{\epsilon_0}{2\delta}(1 - e^{-i\delta t})$$

Floquet map $t = nT = 2\pi n/\Omega$

$$\tilde{a}(nT) = a(nT)$$

Thus

$$a(nT) = a((n-1)T)e^{-i\delta T} - \frac{\epsilon_0}{2\delta}(1 - e^{-i\delta T})$$

for resonance $\delta = 0$

$$a(nT) = a((n-1)T) - i\frac{\epsilon_0 t}{2}$$

Just a displacement in the phase plane

Driven SHO.

Dispalcement operator

$$D(\alpha) = e^{\alpha a^\dagger + \alpha^* a}$$

$$D^\dagger(\alpha) a D(\alpha) = a + \alpha$$

(prove this)

So Floquet map on states is

$$|\psi_{n+1}\rangle = D(\alpha)|\psi_n\rangle$$

$$\alpha = -i\epsilon_0 t/2$$

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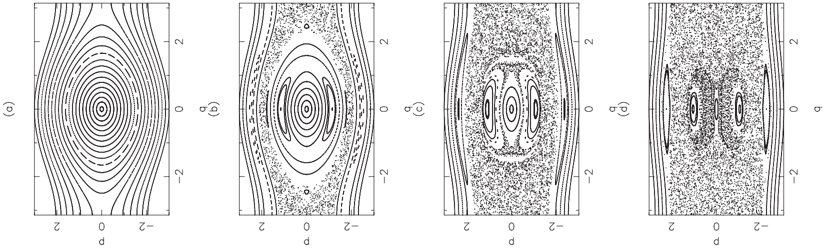

Figure: Plot of classical stroboscopic phase space portraits. (a) $\epsilon = 0.0$, (b)$\epsilon = 0.1$, (c) $\epsilon = 0.2$, (d) $\epsilon = 0.3$.

Particle in a periodic potential.

Floquet eigenstates.

$$|\psi\rangle_n = \hat{F}|\psi\rangle_{n-1}$$

Assume we can find the eigenstates of $\hat{F}$ (not easy!).

$$\hat{F}|\phi_\nu\rangle = e^{-i\phi_\nu}|\phi_\nu\rangle$$

as $\hat{F}$ is unitary, the eigenvalues lie on the unit circle in the complex plane.

The expand the initial state in this basis

$$|\psi\rangle_0 = \sum_\nu c_\nu|\phi_\nu\rangle$$

$$|\psi\rangle_n = \hat{F}^n \sum_\nu c_\nu|\phi_\nu\rangle = \sum_\nu c_\nu e^{-in\phi_\nu}|\phi_\nu\rangle$$

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$$\hat{F}|e_n, p\rangle = \exp\left(-2\pi i\, e_n(p)/\hbar\right)|e_n, p\rangle \; ,$$
$$\hat{S}_{2\pi}|e_n, p\rangle = \exp\left(2\pi i\, p/\hbar\right)|e_n, p\rangle \; .$$

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$e_n(p)$ is called the quasi-energy only defined moduluo $\hbar$.

Particle in a periodic potential.

Given arbitrary initial state, $|\psi\rangle$,

$$\hat{F}^s|\psi\rangle = \sum_n \int_{-\bar{k}/2}^{\bar{k}/2} \exp\left(-2\pi is\, e_n(p)/\bar{k}\right) |e_n, p\rangle\langle e_n, p|\psi\rangle \, dp \ ,$$

after s cycles of the driving term.

Prove that $e_n(-p) = e_n(p)$.

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For $p \in (-\bar{k}/2, 0)$ the new quasi-stationary states

$$|e_n, p\rangle_{\pm} = \frac{1}{\sqrt{2}} \left(|e_n, p\rangle \pm |e_n, -p\rangle \right) \ ,$$

are even $(+)$ and odd $(-)$ parity under $\hat{P}$.

Expanding $|e_n, p\rangle_\pm$ in terms of momentum eigenstates,

$$|e_n, p\rangle_\pm = \sum_{m=-\infty}^{\infty} \frac{a_m}{\sqrt{2}} \left(|-p + m\hbar\rangle \pm |p - m\hbar\rangle \right) \ ,$$

it follows from the anti-linearity of time reversal,

$$\hat{F}\,\hat{T}|e_n, p\rangle_\pm = \exp(-2\pi i e_n(p)/\hbar)\,\hat{T}|e_n, p\rangle_\pm \ .$$

Thus $|e_n, p\rangle_\pm$ and $\hat{T}|e_n, p\rangle_\pm$ are degenerate.

Particle in a periodic potential.

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The operator $\hat{F}$ may be numerically diagonalized for the quasi-momentum $p = 0$. Using the parity and time-reversal symmetries, reduce the problem of diagonalizing a complex unitary operator to an equivalent problem of diagonalizing a real symmetric operator. Then use Householder's method and the QL algorthm to ensure that the numerical quasi-stationary states are real in the momentum basis and strictly orthogonal.

Particle in a periodic potential.

Husimi function for Floquet eigenstates with quasi-momentum $p = 0$, and $\hbar = 0.05$.

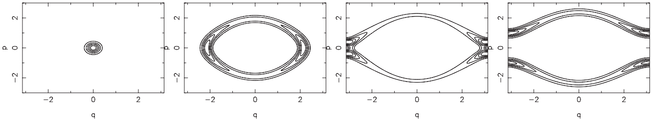

Figure: Some Q functions of Floquet operator when $\epsilon = 0.0$. (a) the ground state, (b) a libration state, (c) a separatrix state, (d) a rotation state.

Particle in a periodic potential.

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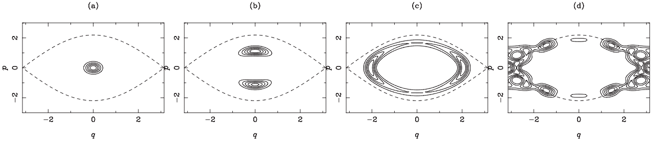

Figure: Some Q functions of Floquet operator eigenstates when $\epsilon = 0.1$.

Particle in a periodic potential.

Husimi function for Floquet eigenstates with quasi-momentum $p = 0$, and $\hbar = 0.05$.

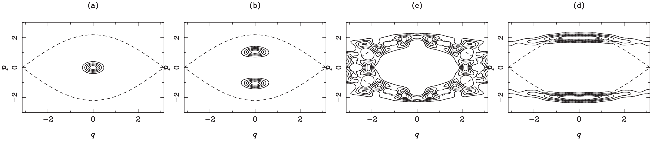

Figure: Some Q functions of Floquet operator eigenstates when $\epsilon = 0.2$.

Particle in a periodic potential.

Husimi function for Floquet eigenstates with quasi-momentum $p = 0$, and $\hbar = 0.05$.

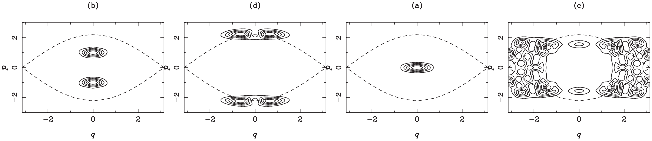

Figure: Some Q functions of Floquet operator eigenstates when $\epsilon = 0.3$.