MATH4104: Quantum nonlinear dynamics. Lecture NINE. Quantum dynamics in a periodic driven systems.

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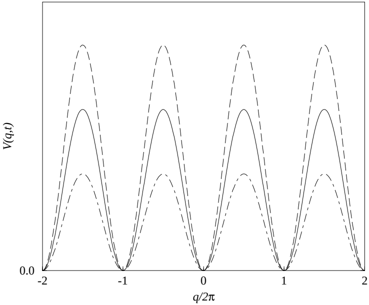
Quantum dynamics on a resonance.

Include a modulation of the potential depth.

$$H(t) = \frac{p^2}{2} - \kappa(1 - 2\epsilon \cos t) \cos(q)$$



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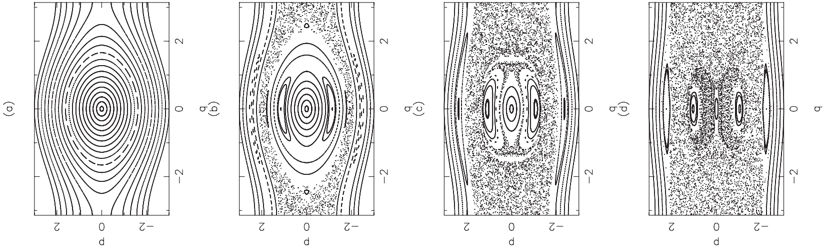


Figure: Plot of classical stroboscopic phase space portraits. (a) $\epsilon = 0.0$, (b)$\epsilon = 0.1$, (c) $\epsilon = 0.2$, (d) $\epsilon = 0.3$.

What is a resonance.

$$H(J, \Theta, t) = H_0(J) + \epsilon H_1(J, \Theta, t)$$

$H_1 = 2\kappa \cos t \cos q$.

$\underline{\epsilon = 0}$.

Canonical transform $(q, p) \rightarrow (J, \Theta)$ such that H_0 is a function of J only and the classical frequency is $\omega_{cl}(J)$

$\underline{\epsilon \neq 0}$
Fourier analyse the first-order (in ϵ) term;

$$H_1 = \sum_{m=0,\ \pm 1,\ \pm 2, \ldots} H_{1,2m} (\exp i(t + 2m\Theta) + c.c.)\ .$$

Then *first-order resonance* solutions occur for J given by

$$2m\, \omega_{cl}(J) - 1 = 0\ , \quad m = 0,\ \pm 1,\ \pm 2, \ldots\ .$$

What is a resonance.

Second-order resonances.
find approximate action-angle variables $(\bar{J}, \bar{\Theta})$ for $H(J, \Theta, t)$ up to and including the order ϵ terms.

if J is not close to a first-order resonance, use a canonical transformation to a new Hamiltonian

$$\bar{H}(\bar{J}, \bar{\Theta}, t) = H_0(\bar{J}) + \epsilon^2 H_2(\bar{J}, \bar{\Theta}, t) + \dots .$$

There is no order ϵ term because H_1 is purely oscillatory.

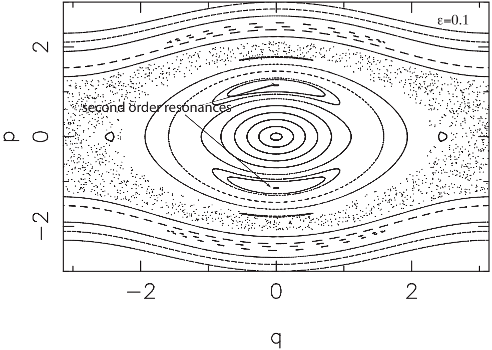
Fourier analysis of the H_2.

$$\begin{aligned} H_2 \quad &= \sum_{m=0,\ \pm1,\ \pm2} H_{2,2m} \left(\exp i(2t + 2m\Theta) + c.c\right) \\ &\quad + \text{time independent terms} . \end{aligned}$$

second-order resonances occur for $\bar{J}$ given by

$$m\omega_{cl}(\bar{J}) - 1 = 0 \ , m = 0,\ \pm1,\ \pm2, \dots$$

The pair of second order resonances.



All the first-order resonances are clustered near the separatrix. There is one second-order resonance $\omega_{cl}(\bar{J}) = \pm 1$ giving two stable fixed points of the stroboscopic map at $(q, p) \approx (0.0 \pm 1.2)$ and the width $\Delta J \approx 0.44$.

Classical dynamics on a second order resonance.

Use a Gaussian state, $(q_0, p_0) = (0, 1.0)$, $\sigma_q = 0.084$ and $\sigma_p = 0.036$, centered on the second-order resonance $(q, p) \approx (0.0, 1.2)$.
Plot the momentum mean $\langle p \rangle$ and variance $V(p)$ as a function of the strobe number s a

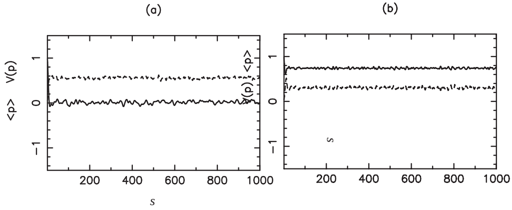


Figure: Plot of classical momentum statistics versus strobe number s. (a) $\epsilon = 0.0$, (b) $\epsilon = 0.1$. Solid line, $\langle p \rangle$; dashed line, $V(p)$.

Classical dynamics on a second order resonance.

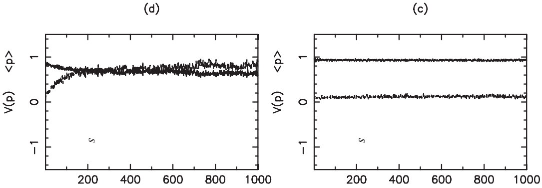


Figure: Plot of classical momentum statistics versus strobe number s. (c) $\epsilon = 0.2$, (d) $\epsilon = 0.3$. Solid line, $\langle p \rangle$; dashed line, $V(p)$.

Classical dynamics on a second order resonance.

As the perturbation parameter ϵ is increased from 0.0 to 0.2

- the growth in momentum variance is suppressed and the mean momentum remains near its initial value indicating that the classical distribution is being localized about the stable fixed point.

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- the growth in momentum variance is suppressed and the mean momentum remains near its initial value indicating that the classical distribution is being localized about the stable fixed point.
- The width of the stable region is increasing linearly with ϵ until the classical distribution is contained within the region of the elliptic fixed point.
- when ϵ is increased to 0.3 the stable region of phase space begins to shrink because of the destruction of KAM tori. As a result we see that the classical distribution becomes delocalized again.

Quantum dynamics on a second order resonance.

We will set $\hbar = 0.05$. To a good approximation the band structure of the driven Hamiltonian can be neglected and restrict to states with the quasi-momentum $p = 0$.

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The Floquet operator $\hat{F}$ was found by numerically integrating the Schrödinger equation in the momentum representation.

$$\hat{H}_a(t) = \frac{\hat{p}^2}{2} + 2\kappa\left(1 - 2\epsilon\cos t\right)\sin^2(\hat{q}/2)\ .$$

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The initial state $|\psi\rangle$ is a minimum-uncertainty state with $\langle\hat{q}\rangle = 0.0$, $\langle\hat{p}\rangle = 1.0$, and $\langle\Delta\hat{p}^2\rangle = 0.01$.

it has a Q function equal to the classical probability distribution used in the classical approximation.

Quantum dynamics on a second order resonance.

$\epsilon = 0$

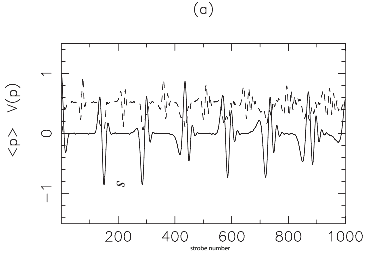


Figure: Solid line, $\langle \hat{p} \rangle$; dashed line, $V(\hat{p})$.

The decrease in momentum variance when s is a multiple of 150 indicates a revival of the initial wave packet due to the quadratic dependence of $E_n(p)$ on n.

Quantum dynamics on a second order resonance.

$$\epsilon = 0.1$$

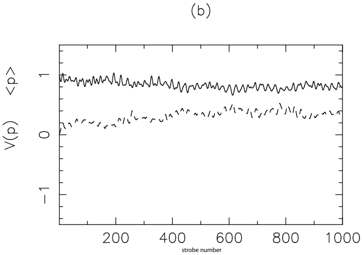


Figure: Solid line, $\langle \hat{p} \rangle$; dashed line, $V(\hat{p})$.

Quantum dynamics on a second order resonance.

$$\epsilon = 0.2$$

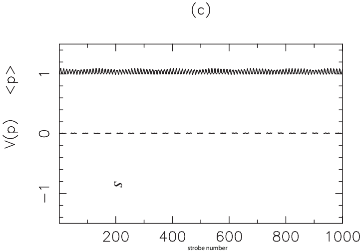


Figure: Solid line, $\langle \hat{p} \rangle$; dashed line, $V(\hat{p})$.

Quantum dynamics on a second order resonance.

$$\epsilon = 0.3$$

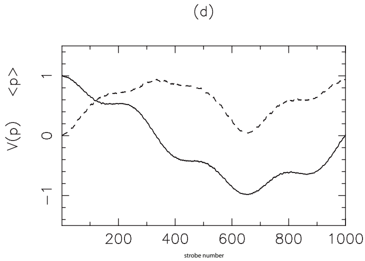


Figure: Solid line, $\langle \hat{p} \rangle$; dashed line, $V(\hat{p})$.

quantum tunneling between fixed points: *dynamical tunneling*.

Quantum dynamics on a second order resonance.

Phasor representation of the probability distribution of the state $|\psi\rangle$ in the $\epsilon = 0.0$ and $\epsilon = 0.2$ bases of quasi-stationary states.

The length of each phasor χ_m equals the overlap probability $|\langle e_n, p|\psi\rangle|^2$ and its angle equals the eigenphase $-2\pi\, e_n(p)/\hbar$.

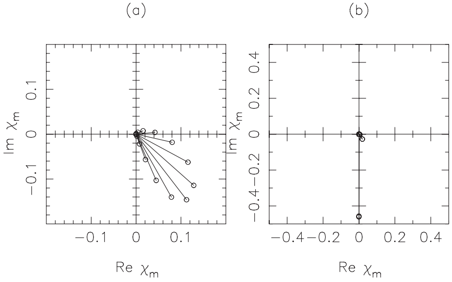


Figure: (a) $\epsilon = 0.0$, (b) $\epsilon = 0.2$.

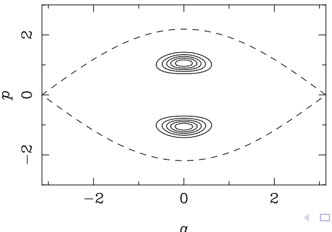
Quantum dynamics on a second order resonance.

Comparing the two distributions we see that as the perturbation parameter is increased the support on the quasistationary states has decreased to two states with almost equal quasifrequencies.

These states have opposite parity under $p \to -p$. Denote as $|e_+\rangle$ and $|e_-\rangle$

The Husimi function for these states are localised on the second order resonances.

(b)



Quantum dynamics on a second order resonance.

When ϵ is increased to 0.3 we would expect the quantum motion to reflect the delocalization of the classical atomic distribution.

Since the minimum uncertainty state is the sum of two quasistationary states with opposite parity we would expect to find that it is now possible for the atom to coherently tunnel between the reonances $(q, p) \approx (0.0, 1.2)$ and its reflected partner $(q, p) \approx (0.0, -1.2)$.

This is due to the detuning of the two dominant quasi-energies.

Quantum dynamics on a second order resonance.

Initial state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|e_+\rangle + |e_-\rangle \right)$$

$$\hat{F}^n|\psi\rangle = \frac{1}{\sqrt{2}} \left(e^{-in\phi_+}|e_+\rangle + e^{-in\phi_-}|e_-\rangle \right)$$

where

$$\phi_\pm = -2\pi\, e_\pm/\hbar$$

When $n(\phi_- - \phi_+) = \pi$, state is

$$\approx |e_+\rangle - |e_-\rangle$$

, which must be localised on the opposite fixed point.

Quantum resonances.

There is a quantum analog of classical second order perturbation theory that leads to the identification of first and second order resonances.

$$\hat{H}(t) = \hat{H}_0 + \epsilon \hat{H}_1(t)\ .$$

For $\epsilon = 0$ the Floquet operator is

$$\hat{F} = \exp\left(\frac{-2\pi i \hat{H}_0}{\hbar}\right)\ ,$$

where $\hat{H}_0 = \hat{p}^2/2 - \kappa \cos \hat{q}$.

Denote the stationary states of $\hat{H}_0$ with energy $E_n(p)$ by $|E_n, p\rangle$, then $|E_n, p\rangle$ is a quasi-stationary state for $\hat{F}$ with quasi-energy $e_n(p) = E_n(p)$.

Quantum resonances.

In analogy with time-independent quantum perturbation theory we assume that for small ϵ the perturbed quasistationary states $|e_n, p\rangle$ and quasifrequency $e_n(p)$ are close to $|E_n, p\rangle$ and $E_n(p)$ respectively, and then attempt to find corresponding asymptotic expansions in ϵ.

Quantum resonances.

Let $|e_n, p, t\rangle$ satisfy the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt}|e_n, p, t\rangle = \hat{H}(t)|e_n, p, t\rangle$$

subject to the condition $|e_n, p, 2\pi\rangle = \exp(-2\pi i e_n(p)/\hbar)|e_n, p, 0\rangle$.

Define, $|v_n, p\rangle = \exp(ie_n(p)t/\hbar)|e_n, p, t\rangle$ which are periodic in t, and

$$e_n(p)|v_n, p\rangle = -i\hbar \frac{d}{dt}|v_n, p\rangle + \hat{H}(t)|v_n, p\rangle \ .$$

Quantum resonances.

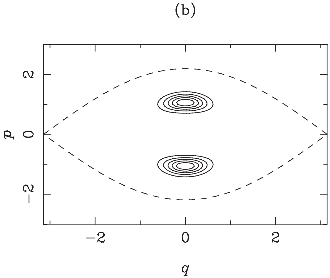
Seek,

$$\begin{aligned} e_n(p) &= E_n(p) + \epsilon e_n^{(1)}(p) + \epsilon^2 e_n^{(2)}(p) + O(\epsilon^3)\ , \\ |v_n, p\rangle &= |E_n, p, 0\rangle + \epsilon |v_n^{(1)}, p\rangle + \epsilon^2 |v_n^{(2)}, p\rangle + O(\epsilon^3) \quad . \end{aligned}$$

As in classical p.t. singularities arise in these expansions at *resonances*.

For each classical resonance $\Delta n\, \omega_{cl}(J) - l = 0$ there will be energy bands with $E_n(p)$ and $E_{n-\Delta n}(p)$ satisfying the near-resonance condition $E_n(p) - E_{n-\Delta n}(p) \approx \hbar l$. The perturbed quasistationary state $|e_n, p\rangle$ will rapidly develop a significant component along $|e_{n-\Delta n}, p\rangle$ as ϵ is increased.

Quantum resonances.



This Floquet state is a superposition of a dominant state $|E_{11}, 0\rangle$ and two other states $|E_9, 0\rangle$ and $|E_{13}, 0\rangle$ satisfying the near-resonant conditions: $E_{11}(0) - E_9(0) = 0.103 \approx 2\hbar$, and $E_{11}(0) - E_{13}(0) = -0.101 \approx -2\hbar$. The interference between near resonant states has caused the Q function in the figure to become concentrated about the stable regions of the classical second-order resonance.