Q1. (a) Derive Hamilton's equations of motion from Lagrange's equations of motion.

(3 marks)

Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{where } L(q_i, \dot{q}_i, t).$$

Now define the conjugate momenta

$p_i = \frac{\partial L}{\partial \dot{q}_i}$ and there equations become

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \qquad (1)$$

If $H(q_i, p_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i(q_i, p_i, t), t)$

then $$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} + p_j \frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q_i} = -\frac{\partial L}{\partial q_i} \qquad (2)$$

Also $$\frac{\partial H}{\partial p_i} = \dot{q}_i + p_i \frac{\partial \dot{q}_j}{\partial p_i} - \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_i} = \dot{q}_i \qquad (3)$$

$$\left.\begin{aligned} (3) &\Rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i} \\ (1) + (2) &\Rightarrow \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{aligned}\right\} \text{Hamiton's eqns of motion.}$$

Question 1 continued on next page.

Q1. (b) Show that the following Hamiltonians are integrable and give their integrals of motion.

(i)
$$H(q, p, t) = \frac{p^2}{2} + \cos(q) + e^{-t}$$

(ii)
$$H(\theta_1, \theta_2, I_1, I_2) = I_1^2 + 2I_1I_2 + I_2^2 + I_1I_2\cos(\theta_1 - 3\theta_2)$$

(7 marks)

i) Since this is a 1 dof system integrability is ensured if there is one constant of the motion. Here that is $\bar{H} = p^2/2 + \cos(q)$ which is an equally good Hamiltonian for the system since e^{-t} is a complete differential.

Alternatively use a generating function
$$F = qP + e^{-t}$$
to transform to "new" coordinates $P = p$, $Q = q$ with new Hamiltonian $\bar{H} = H + \frac{\partial F}{\partial t}$.

ii) Transform to new conjugate variables with
$$\phi_1 = \theta_1 - 3\theta_2 , \quad \phi_2 = \theta_2 \quad \text{via.}$$
$$F_2(\theta_i, J_i) = J_1(\theta_1 - 3\theta_2) + J_2\theta_2 \Rightarrow I_1 = J_1, I_2 = -3J_1 + J_2$$

The Hamiltonian
$$H(\phi_i, J_i) = H_0(J_i) + J_1(J_2 - 3J_1)\cos\phi_1$$
is now independent of $\phi_2 \Rightarrow \frac{\partial H}{\partial \phi_2} = 0 \Rightarrow J_2$ constant of motion.

So, for this 2 dof system, there are 2 constants of the motion : H and $J_2 = I_2 + 3I_1$

Question 2 on next page.

$\Rightarrow$ integrable.

(Actually $\{J_2, H\} = 0 \Rightarrow$ completely integrable.)

Q2. Give the primary resonance conditions for the following Hamiltonian.

$$H(\theta_1,\, \theta_2,\, I_1,\, I_2) = 2I_1^2 + I_2^2 + \epsilon(I_1^2 \sin(\theta_1)^2 \cos(\theta_2) + I_2^2)$$

If H is fixed as 1, which tori of the unperturbed system, that is the system with $\epsilon = 0$, will be resonant?

Assuming you are not close to resonance give the near identity generating function that removes all the oscillating terms to order ϵ. Hence give the approximate integrals of motion in the original variables.

(10 marks)

Since $\sin^2\theta_1 \cos\theta_2 = \dfrac{\cos\theta_2}{2} - \dfrac{(\cos(2\theta_1+\theta_2) + \cos(2\theta_1-\theta_2))}{4}$

There are three resonance conditions (primary)

$$\frac{\partial H_0}{\partial I_2} = 0 \;, \quad 2\frac{\partial H_0}{\partial I_1} + \frac{\partial H_0}{\partial I_2} = 0 \quad \text{or} \quad 2\frac{\partial H_0}{\partial I_1} - \frac{\partial H_0}{\partial I_2} = 0$$

$$\Rightarrow \quad I_2 = 0 \;, \quad 4I_1 + 2I_2 = 0 \quad \text{or} \quad 4I_1 - 2I_2 = 0 \,.$$

If $H_0 = 2I_1^2 + I_2^2 = 1$ then these imply that

$$\left\{I_1 = \tfrac{\pm 1}{\sqrt{2}},\, I_2 = 0\right\},\quad \left\{I_1 = \tfrac{\pm 1}{\sqrt{6}},\, I_2 = \mp\tfrac{2}{\sqrt{6}}\right\},\quad \left\{I_1 = \tfrac{\pm 1}{\sqrt{6}},\, I_2 = \tfrac{\pm 2}{\sqrt{6}}\right\}$$

Usually the action is taken as >0

$$\Rightarrow \left\{I_1 = \tfrac{1}{\sqrt{2}},\, I_2 = 0\right\},\ \text{No soln},\ \left\{I_1 = \tfrac{1}{\sqrt{6}},\, I_2 = \tfrac{2}{\sqrt{6}}\right\}.$$

more space for question 2 over page.

Space for question 2

$$H = 2I_1^2 + I_2^2 + \epsilon\left(I_2^2 + \frac{I_1^2}{4}\left(2\cos\theta_2 - \cos(2\theta_1-\theta_2) - \cos(2\theta+\theta_2)\right)\right)$$

Use a near identity generating function

$$F_2(\theta_i, J_i) = \theta_1 J_1 + \theta_2 J_2 + \epsilon\left(g_1 \sin\theta_2 + g_2 \sin(2\theta_1-\theta_2) + g_3 \sin(2\theta_1+\theta_2)\right)$$

to remove all the oscillating terms to order ϵ, where $g_i(J_i)$

$$\Rightarrow \quad I_1 = J_1 + \epsilon\left(g_2 \cos(2\theta_1-\theta_2) - g_3\cos(2\theta_1+\theta_2)\right)$$

$$I_2 = J_2 + \epsilon\left(g_1\cos\theta_2 + 2g_2\cos(2\theta_1-\theta_2) + 2g_3\cos(2\theta_1+\theta_2)\right)$$

To order ϵ the Hamiltonian becomes.

$$H = 2J_1^2 + J_2^2 - \epsilon 4J_1\left(g_2\cos(\ ) + g_3(\ )\right) + \epsilon 2J_2\left(g_1\cos\theta_2 + 2g_2\cos(\ ) + 2g_3\cos(\ )\right) + \epsilon\left(J_2^2 + \frac{J_1^2}{4}\left(2\cos\theta_2 - \cos(\ ) - \cos(\ )\right)\right)$$

So choose $2g_1J_2 = -\frac{J_1^2}{2}$ $\qquad \underline{\underline{g_1 = \frac{J_1^2}{4J_2}}}$

$$(-4J_1 + 4J_2)g_2 = \frac{J_1^2}{4} \Rightarrow \underline{\underline{g_2 = \frac{J_1^2}{16(J_2-J_1)}}}$$

$$4(J_1+J_2)g_3 = \frac{J_1^2}{4} \Rightarrow \underline{\underline{g_3 = \frac{J_1^2}{16(J_1+J_2)}}}$$

The approximate integrals of motion are J_1 and J_2

$$J_1 = I_1 + \epsilon\left(\frac{I_1^2}{16(I_2-I_1)}\cos(2\phi_1-\phi_2) - \frac{I_1^2}{16(I_2+I_1)}\cos(2\phi_1+\phi_2)\right)$$

Question 3 see next page.

$$J_2 = I_2 + \epsilon\left(\frac{I_1^2}{4I_2}\cos\phi_2 + \frac{2I_1^2}{16(I_2-I_1)}\cos(2\phi_1-\phi_2) + \frac{2I_1^2}{16(I_2-I_1)}\cos(2\phi_1+\phi_2)\right)$$

Q3. (a) Describe what happens as you perturb the following twist map

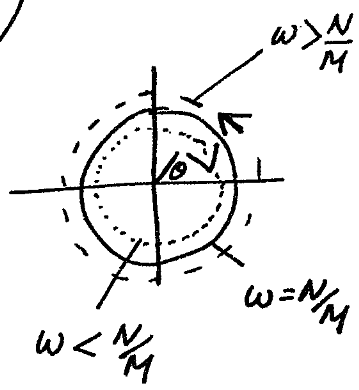
$$I_{n+1} = I_n$$

$$\theta_{n+1} = \theta_n + 2\pi\omega(I_n), \quad mod(1)$$

if $\omega = \frac{N}{M}$, given that $\omega(I_n)$ is continuous function and the map remains area preserving. **(5 marks)**

If the unperturbed map is T

$$\begin{pmatrix}\theta_{n+1} \\ I_{n+1}\end{pmatrix} = T\begin{pmatrix}\theta_n \\ I_n\end{pmatrix} = \begin{pmatrix}\theta_n + 2\pi\omega(I_n) \\ I_n\end{pmatrix}$$



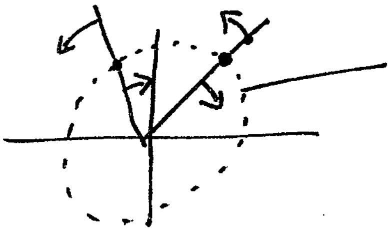
Then for $\omega = N/M$ T^M is the identity.

If $\omega > N/M$ then T^M twists anticlockwise

But for $\omega < N/M$ T^M twists clockwise

Now Perturb this to T_ϵ. The relative twists will be preserved, ie further out the twist is anticlockwise but further in " " " clockwise.

So on each radial line there is a point which is not twisted!



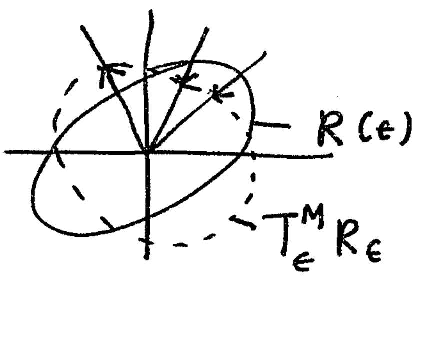
Let the curve of these points be R_ϵ

Consider $T_\epsilon^M R_\epsilon$, each point moves radially.

But area is preserved so

$$T_\epsilon^M R_\epsilon \cap R_\epsilon \neq \phi$$

infact they must intersect in an even no of points.



Furthere if x is an intersection point then so is $T_\epsilon x \Rightarrow \exists$ at least $2kM$ ~~period~~ fixed points of T_ϵ^M or period-M points of T_ϵ.

Question 3 continued on next page.

Q3. (b) Show that the following map

$$u_{n+1} = u_n - A\sin(2\pi\phi_n)$$

$$\phi_{n+1} = \phi_n + (u_{n+1})^2 \quad mod(1)$$

is a product of two involutions and find any lines or curves which are fixed under the involutions of the map. Using this or otherwise investigate the possibility of symmetric period-2 points where one iterate lies on the line $\phi_n = 0$.

(5 marks)

First find the involutions:

$$I_1\begin{pmatrix}\phi\\u\end{pmatrix} = \begin{pmatrix}-\phi\\u - A\sin 2\pi\phi\end{pmatrix} \text{ then } I_1^2\begin{pmatrix}\phi\\u\end{pmatrix} = \begin{pmatrix}-(-\phi)\\u - A\sin 2\pi(\phi) - A\sin 2\pi(-\phi)\end{pmatrix} = \begin{pmatrix}\phi\\u\end{pmatrix}$$

$$I_2\begin{pmatrix}\phi\\u\end{pmatrix} = \begin{pmatrix}-\phi + u^2\\u\end{pmatrix} \text{ then } I_2^2\begin{pmatrix}\phi\\u\end{pmatrix} = \begin{pmatrix}-(-\phi+u^2)+u^2\\u\end{pmatrix} = \begin{pmatrix}\phi\\u\end{pmatrix}$$

such that $I_2 I_1$ is the map:

$$I_2 I_1\begin{pmatrix}\phi\\u\end{pmatrix} = \begin{pmatrix}-(-\phi)+(u - A\sin 2\pi\phi)^2\\u - A\sin 2\pi\phi\end{pmatrix} = \begin{pmatrix}\phi + (u - A\sin 2\pi\phi)^2\\u - A\sin 2\pi\phi\end{pmatrix}.$$

Fixed points of I_1

$\phi = -\phi$ and $u = u - A\sin 2\pi\phi \Rightarrow \phi = 0$ or $\frac{1}{2}$ are fixed lines

Fixed points of I_2

$\phi = -\phi + u^2 \Rightarrow u^2 = 2\phi \text{ mod } 1 \Rightarrow \left.\begin{array}{l}\phi = \frac{u^2}{2}\\ \phi = \frac{1}{2} + \frac{u^2}{2}\end{array}\right\}$ fixed curves.

There is a symmetric period-2 point at

$u_n = \frac{1}{\sqrt{2}}$, $\phi_n = 0 \Rightarrow u_{n+1} = \frac{1}{\sqrt{2}}$ $\phi_{n+1} = \frac{1}{2}$

$u_{n+2} = \frac{1}{\sqrt{2}}$ $\phi_{n+2} = 1/_{mod(1)} = 0.$

There are various ways to get this;

Question 4 see next page.

directly by considering period-2 pts with $\phi_n = 0$, or by using properties of involutions.