INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

CANDIDATES MUST NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Second Semester Examination, November, 2005

MATH4104

ADVANCED HAMILTONIAN DYNAMICS & CHAOS

(Unit Courses, Inf. Tech.)

Time: **THREE HOURS** for working

Ten minutes for perusal before examination begins

Check that this examination paper has 15 printed pages!

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION PAPER!
Students should attempt all questions.
The questions each carry 10 marks and part marks are as indicated.
The exam paper is a total of 60 marks.
Calculators allowed.

FAMILY NAME (PRINT): ______________________

GIVEN NAMES (PRINT): ______________________

STUDENT NUMBER: | | | | | | | | |

SIGNATURE: ______________________

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
1		4	
2		5	
3		6	
TOTAL MARKS			

Q1. (a) Find the function that is related to

$$L(r,\,\theta,\,\phi,\,\dot{\theta},\,\dot{\phi}) = \frac{mr^2}{2}(\dot{\theta}^2 + \dot{\phi}^2)\sin^2\theta - \frac{A}{r}$$

by a Legendre transform with both $\dot{\theta}$ and $\dot{\phi}$ active and the other variables passive.

Show that the Legendre transform relations hold for your solution.

(4 marks)

Question 1 continued on next page.

Q1.

(b) Show carefully that the following Hamiltonians are integrable and give their integrals of motion.

(i) $H(q,\, p,\, t) = \dfrac{p^2}{2} + \cos(q + \omega t)\cos(q + 2\omega t)$

(ii) $H(\theta_1,\, \theta_2,\, I_1,\, I_2) = 2I_1^2 + I_2^2 + I_1 I_2 \cos(3\theta_1 - \theta_2)$

(6 marks)

Question 2 on next page.

Q2. (a) Give the primary resonance conditions for the following Hamiltonian.

$$H(\theta_1,\, \theta_2,\, I_1,\, I_2) = 2I_1^2 + 3I_1 I_2 + 4\epsilon I_1^2 \sin(\theta_2)\cos(\theta_2)\cos(\theta_1)$$

(4 marks)

Question 2 continued on next page.

Q2.

(b) Construct the near identity transformation that transforms the Hamiltonian in part a) to

$$K(\phi_1,\, \phi_2,\, J_1,\, J_2) = 2J_1^2 + 3J_1J_2 + \epsilon J_1^2 \sin(2\phi_2 - \phi_1)$$

to first order in ϵ. Give the actual functions $\phi_i(\theta_i,\, I_i)$ and $J_i(\theta_i,\, I_i)$ to first order in ϵ.

(6 marks)

more space for question 2 over page.

Space for question 2

Question 3 see next page.

Q3. (a) Give the continued fraction expansion of $\dfrac{47-\sqrt{5}}{38}$.

(3 marks)

Question 3 continued on next page.

Q3.

(b) Derive conditions on $f(\theta_n,\, I_{n+1})$ and $g(\theta_n,\, I_{n+1})$ for the following twist map to evolve canonically.

$$I_{n+1} = I_n + \epsilon f(\theta_n,\, I_{n+1})$$
$$\theta_{n+1} = \theta_n + 2\pi\omega(I_{n+1}) + \epsilon g(\theta_n,\, I_{n+1}), \qquad mod(2\pi)$$

(3 marks)

Question 3 continued on next page.

Q3.

(c) Show that the Standard Map

$$I_{n+1} = I_n - K \sin \theta_n$$

$$\theta_{n+1} = \theta_n + I_{n+1}$$

is a product of two involutions. **(4 marks)**

Question 4 see next page.

Q4 What is the BGS conjecture ? In less than one page explain what is has to say about the quantum description of systems that are classically chaotic.

(10 marks)

Space for question 4

Question 5 on next page.

Q5 Consider a system with integrable (regular) classical dynamics. Suppose the corresponding quantum system is described by an N dimensional Hilbert space. Assume that the dynamics of this system is well described by a suitable random matrix ensemble and order the energy levels so that $\epsilon_1 \leq \epsilon_2 \ldots \leq \epsilon_N$. Let $p_r(\lambda)$ be the probability of obtaining an energy level spacing of λ for the r^{th}-nearest neighbour spacing (that is to say the energy level spacing $\epsilon_{n+r} - \epsilon_n = \lambda$ for arbitrary n) . Show that for systems with a regular dynamics this is given by

$$p_r(\lambda) = \frac{\lambda^{r-1}}{(r-1)!} e^{-\lambda}$$

assuming we have scaled the energy to give an average level spacing of unity.

(10 marks)

space for question 5 on next page.

Space for question 5

Question 6 on next page.

Q6 Consider a quantum system described by a Hilbert space of dimension N. Let $|\psi_0\rangle$ be an arbitrary initial state. The survival probability, $P_{\psi_0}(t)$, is defined as the probability that the system will still be found in its initial state, $|\psi_0\rangle$, after a time $t > 0$. If we average over all initial states and over a suitable random matrix ensemble it is possible to write the average survival probability as

$$\overline{P(t)} = \frac{2}{N+1} + \frac{N-1}{N+1}\int_0^\infty P^N(\lambda)\cos(\lambda t)d\lambda$$

where $P^N(\lambda)$ is the probability that an energy level spacing of λ occurs between any pair of energy levels.

(a) Show that, in general,

$$P^N(\lambda) = \frac{2}{N(N-1)}\sum_{r=1}^{N-1}(N-r)p_r(\lambda)$$

where $p_r(\lambda)$ is the probability of obtaining an energy level spacing of λ for the r^{th}-nearest neighbour spacing

(2 marks)

(b) Prove that, for a system with regular dynamics the average survival probability, $\overline{P(t)}$, never drops below its long time limit of $2/(N+1)$, while for chaotic systems it may drop below the long time limit for some time.

(8 marks)

space for question 6 on next page.

Space for question 6