Q1. (a) Find the function that is related to

$$L(r, \theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{mr^2}{2}(\dot{\theta}^2 + \dot{\phi}^2)\sin^2\theta - \frac{A}{r}$$

by a Legendre transform with both $\dot{\theta}$ and $\dot{\phi}$ active and the other variables passive.
Show that the Legendre transform relations hold for your solution.

(4 marks)

First define the conjugate momenta:

$$p_r = \frac{\partial L}{\partial \dot{r}} = 0 \;,\; p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\sin^2\theta,\; p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta$$

Then the Hamiltonian, which is the function related to L by the Legendre transform is

$$H(r,\theta,\phi,p_\theta,p_\phi) + L(r,\theta,\phi,\dot{\theta},\dot{\phi}) = \dot{\theta}p_\theta + \dot{\phi}p_\phi$$

$$\Rightarrow H(r,\theta,\phi,p_\theta,p_\phi) = \frac{(p_\theta^2 + p_\phi^2)}{2mr^2\sin^2\theta} + \frac{A}{r}$$

Legendre transform relations

Active variables

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2\sin^2\theta} \checkmark \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta} \checkmark$$

passive variables

$$\frac{\partial L}{\partial r} + \frac{\partial H}{\partial r} = \frac{A}{r^2} - \frac{A}{r^2} = 0 \checkmark \qquad \frac{\partial L}{\partial \theta} + \frac{\partial H}{\partial \theta} = mr^2(\dot{\theta}^2 + \dot{\phi}^2)\sin\theta\cos\theta - \frac{(p_\theta^2 + p_\phi^2)\cos\theta}{mr^2\sin^3\theta} = 0 \checkmark$$

$$\frac{\partial L}{\partial \phi} + \frac{\partial H}{\partial \phi} = 0 + 0 = 0 \checkmark$$

Question 1 continued on next page.

Q1.

~~(b) Show carefully that the following~~ Hamiltonians are integrable and give their integrals of motion.

(i) $H(q, p, t) = \frac{p^2}{2} + \cos(q + \omega t)\cos(q + 2\omega t)$

(ii) $H(\theta_1, \theta_2, I_1, I_2) = 2I_1^2 + I_2^2 + I_1 I_2 \cos(3\theta_1 - \theta_2)$

(6 marks)

i) 1 degree of Freedom System - need 1 constant of the motion.

Since $\cos(q+\omega t)\cos(q+2\omega t) = \frac{1}{2}(\cos(2q+3\omega t) + \cos\omega t)$

$$H(q,p,t) = \frac{p^2}{2} + \frac{1}{2}\cos(2q+3\omega t) + \frac{1}{2}\cos\omega t$$

Move to a rotating frame

$\Rightarrow$ let $Q = q + \frac{3\omega}{2}t$

And remove to purely time dependent term $\frac{\cos\omega t}{2}$

Let $F_2 = P\left(q + \frac{3\omega t}{2}\right) - \int \frac{\cos\omega t}{2}\,dt$

Then new canonical variables $p = P$, $Q = q + \frac{3\omega t}{2}$

and Hamiltonian $K(Q,P) = \frac{P^2}{2} + \frac{1}{2}\cos 2Q + \frac{\partial F_2}{\partial t} + \frac{\cos\omega t}{2}$

$$= \frac{P^2}{2} + \frac{\cos 2Q}{2} + \frac{3P\omega}{2}$$

So $\underline{K = \frac{p^2}{2} + \frac{\cos(2q+3\omega t)}{2} + \frac{3p\omega}{2}}$ is a constant of the motion.

ii) 2 dof system, but time independent $\Rightarrow$ $\underline{H \text{ constant of motion}}$

Move to new variables $\phi_1 = \theta_1 - \theta_2/3$ $\quad \phi_2 = \theta_2$ where

$F_2 = J_1(\theta_1 - \theta_2/3) + J_2\theta_2 \Rightarrow I_1 = J_1$

$\quad I_2 = J_1/3 + J_2$

$H(\phi_1, \phi_2, J_1, J_2) = 2J_1^2 + (J_1/3 + J_2)^2 + J_1(J_1/3 + J_2)\cos 3\phi_1$

is not a function of $\phi_2 \Rightarrow \frac{\partial H}{\partial \phi_2} = 0 \Rightarrow J_2$ is a constant of motion

$\Rightarrow \left(I_2 - \frac{I_1}{3}\right)$ is 2nd constant of motion.

$\Rightarrow$ integrable.

Question 2 on next page.

TURN OVER

Q2. (a) Give the primary resonance conditions for the following Hamiltonian.

$$H(\theta_1, \theta_2, I_1, I_2) = 2I_1^2 + 3I_1I_2 + 4\epsilon I_1^2 \sin(\theta_2)\cos(\theta_2)\cos(\theta_1)$$

(4 marks)

First note that

$$2\sin\theta_2\cos\theta_2 = \sin 2\theta_2$$

and $$4\sin\theta_2\cos\theta_2\cos\theta_1 = \sin(2\theta_2+\theta_1) + \sin(2\theta_2-\theta_1)$$

Now the primary resonance conditions for

$$H = 2I_1^2 + 3I_1I_2 + \epsilon I_1^2(\sin(2\theta_2+\theta_1) + \sin(2\theta_2-\theta_1))$$

are $$2\frac{\partial H_0}{\partial I_2} + \frac{\partial H_0}{\partial I_1} = 0 \quad \text{and} \quad 2\frac{\partial H_0}{\partial I_2} - \frac{\partial H_0}{\partial I_1} = 0$$

Here $$H_0 = 2I_1^2 + 3I_1I_2 \Rightarrow \frac{\partial H_0}{\partial I_1} = 4I_1 + 3I_2$$

$$\frac{\partial H_0}{\partial I_2} = 3I_1$$

$\Rightarrow$ Primary resonance conditions are

$$4I_1 + 3I_2 = \pm 6I_1$$

$$\Rightarrow \quad 3I_2 = 2I_1 \quad \text{or} \quad 10I_1 + 3I_2 = 0$$

Question 2 continued on next page.

Q2.

(b) Construct the near identuty transformation that transforms the Hamiltonian in part a) to

$$K(\phi_1, \phi_2, J_1, J_2) = 2J_1^2 + 3J_1J_2 + \epsilon J_1^2 \sin(2\phi_2 - \phi_1)$$

to first order in ϵ. Give the actual functions $\phi_i(\theta_i, I_i)$ and $J_i(\theta_i, I_i)$ to first order in ϵ.

(6 marks)

let the near identity transformation be

$$F_2(\theta_1, \theta_2, J_1, J_2) = \theta_1 J_1 + \theta_2 J_2 + \epsilon\, g(J_1, J_2) \cos(2\theta_2 + \theta_1)$$

then $I_1 = J_1 - \epsilon g \sin(2\theta_2 + \theta_1)$ $\quad \phi_1 = \theta_1 + \epsilon \frac{\partial g}{\partial I_1} \cos(\ )$

$I_2 = J_2 - \epsilon 2g \sin(2\theta_2 + \theta_1)$ $\quad \phi_2 = \theta_2 + \epsilon \frac{\partial g}{\partial I_2} \cos(\ )$

$$H(\theta_1, \theta_2, I_1(J_i, \theta_i), I_2(J_i, \theta_i))$$

$$= 2(J_1 - \epsilon g \sin(2\theta_2 + \theta_1))^2 + 3(J_1 - \epsilon g \sin(2\theta_2 + \theta_1))(J_2 - \epsilon 2g \sin(\ ))$$

$$+ \epsilon J_1^2 (\sin(2\theta_2 + \theta_1) + \quad \sin(2\theta_2 - \theta_1)) + O(\epsilon^2)$$

$$= 2J_1^2 + 3J_1J_2 + \epsilon(-4J_1 g - 3(2J_1 + J_2)g + J_1^2) \sin(\ )$$

$$+ \epsilon J_1^2 \sin(2\theta_2 - \theta_1) + O(\epsilon^2)$$

So choose

$$(10J_1 + 3J_2) g = J_1^2 \Rightarrow g = \frac{J_1^2}{10J_1 + 3J_2}$$

$\Rightarrow$ to first order in ϵ

$$J_1 = I_1 + \epsilon \frac{I_1^2 \sin(2\theta_2 + \theta_1)}{10I_1 + \epsilon I_2}$$

$$\phi_1 = \theta_1 + \epsilon \frac{10I_1^2 + 6I_1I_2}{(10I_1 + 3I_2)^2} \cos(2\theta_2 + \theta_1)$$

more space for question 2 over page.

Space for question 2

$$J_2 = I_2 + \frac{2\epsilon I_1^2}{(10I_1+3I_2)} \sin(2\theta_2+\theta_1)$$

$$\phi_2 = \theta_2 - \frac{3\epsilon I_1^2}{(10I_1+3I_2)^2} \cos(2\theta_2+\theta_1)$$

Question 3 see next page.

Q3. (a) Give the continued fraction expansion of $\frac{47-\sqrt{5}}{38}$.

~~(3 marks)~~

$$a_0 = \text{integer part of } \frac{47-\sqrt{5}}{38} = 1$$

$$w_1 = \frac{1}{\frac{47-\sqrt{5}}{38} - 1} = \frac{38}{9-\sqrt{5}} = \frac{9+\sqrt{5}}{2}$$

$$\Rightarrow a_1 = 5 \text{ and } w_2 = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}$$

$$a_2 = 1 \quad \therefore \quad w_3 = \frac{2}{\sqrt{5}-1} \quad \text{etc}$$

$$a_n = 1 \text{ for } n \geq 2$$

$$\frac{47-\sqrt{5}}{38} = [1, 5, 1, 1, 1 \ldots]$$

Question 3 continued on next page.

Q3.

(b) Derive conditions on $f(\theta_n, I_{n+1})$ and $g(\theta_n, I_{n+1})$ for the following twist map to evolve canonically.

$$I_{n+1} = I_n + \epsilon f(\theta_n, I_{n+1})$$
$$\theta_{n+1} = \theta_n + 2\pi\omega(I_{n+1}) + \epsilon g(\theta_n, I_{n+1}), \quad mod(2\pi)$$

(3 marks)

Construct a generating function

$F_2(\theta_n, I_{n+1})$ s.t $I_n = \dfrac{\partial F_2}{\partial \theta_n}$ & $\theta_{n+1} = \dfrac{\partial F_2}{\partial I_{n+1}}$

$$\Rightarrow \frac{\partial F_2}{\partial \theta_n} = I_{n+1} - \epsilon f(\theta_n, I_{n+1})$$

$$\frac{\partial F_2}{\partial I_{n+1}} = \theta_n + 2\pi\omega(I_{n+1}) + \epsilon g(\theta_n, I_{n+1})$$

So $F_2 = \theta_n I_{n+1} + 2\pi \int \omega(I_{n+1})\, dI_{n+1} + \epsilon h(\theta_n, I_{n+1})$

where $\dfrac{\partial h}{\partial \theta_n} = -f$ and $\dfrac{\partial h}{\partial I_{n+1}} = g$

$$\Rightarrow \underline{\frac{\partial f}{\partial I_{n+1}} + \frac{\partial g}{\partial \theta_n} = 0}$$

Question 3 continued on next page.

Q3.

(c) Show that the Standard Map

$$I_{n+1} = I_n - K\sin\theta_n$$

$$\theta_{n+1} = \theta_n + I_{n+1}$$

is a product of two involutions. (4 marks)

if $\quad \mathcal{I}_1\begin{pmatrix} I \\ \theta \end{pmatrix} = \begin{pmatrix} I - k\sin\theta \\ -\theta \end{pmatrix}, \quad \mathcal{I}_2\begin{pmatrix} I \\ \theta \end{pmatrix} = \begin{pmatrix} I \\ -\theta + I \end{pmatrix}$

then $\mathcal{I}_2\mathcal{I}_1\begin{pmatrix} I \\ \theta \end{pmatrix} = \mathcal{I}_2\begin{pmatrix} I - k\sin\theta \\ -\theta \end{pmatrix} = \begin{pmatrix} I - k\sin\theta \\ \theta + I - k\sin\theta \end{pmatrix}$

map as required

Also $\mathcal{I}_1^2\begin{pmatrix} I \\ \theta \end{pmatrix} = \mathcal{I}_1\begin{pmatrix} I - k\sin\theta \\ -\theta \end{pmatrix} = \begin{pmatrix} I - k\sin\theta - k\sin(-\theta) \\ -(-\theta) \end{pmatrix}$

$$= \begin{pmatrix} I \\ \theta \end{pmatrix}$$

and $\mathcal{I}_2^2\begin{pmatrix} I \\ \theta \end{pmatrix} = \mathcal{I}_2\begin{pmatrix} I \\ -\theta + I \end{pmatrix} = \begin{pmatrix} I \\ -(-\theta + I) + I \end{pmatrix}$

$$= \begin{pmatrix} I \\ \theta \end{pmatrix}$$

So that $\mathcal{I}_1^2$ and $\mathcal{I}_2^2$ are the identity.
$\Rightarrow$ they are involutions.

Question 4 see next page.