Q1.(a) Find the Hamiltonian $H(x, y, p_x, p_y)$ that is related to the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \left(1 - \sqrt{1 - \dot{x}^2 - \dot{y}^2}\right) - V(x, y)$$

by a **Legendre transform** with both $\dot{x}$ and $\dot{y}$ as active variables and x and y as passive variables.
Then use the Legendre transform relations and Lagranges equations of motion to derive **Hamilton's equations of motion** for the Hamiltonian above. **(6 marks)**

Let $p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{1-\dot{x}^2-\dot{y}^2}}$, $p_y = \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1-\dot{x}^2-\dot{y}^2}}$

then $H(x,y,p_x,p_y) = \dot{x}p_x + \dot{y}p_y - L$ expressed as a function of (x,y,p_x,p_y)

Now $\dot{x}^2 = p_x^2(1-\dot{x}^2-\dot{y}^2)$ and $\dot{y}^2 = p_y^2(1-\dot{x}^2-\dot{y}^2)$

$$\Rightarrow (\dot{x}^2+\dot{y}^2) = \frac{(p_x^2+p_y^2)}{(1+p_x^2+p_y^2)}$$

$$\Rightarrow \dot{x}^2 = \frac{p_x^2}{(1+p_x^2+p_y^2)}, \quad \dot{y}^2 = \frac{p_y^2}{(1+p_x^2+p_y^2)}$$

$$H = \frac{p_x^2+p_y^2}{\sqrt{(1+p_x^2+p_y^2)}} - 1 + \frac{1}{\sqrt{1+p_x^2+p_y^2}} + V(x,y)$$

$$= \sqrt{1+p_x^2+p_y^2} - 1 + V(x,y)$$

From the Legendre Transform $\Rightarrow \dot{x} = \frac{\partial H}{\partial p_x}$, $\dot{y} = \frac{\partial H}{\partial p_y}$

$$\dot{x} = \frac{p_x}{\sqrt{1+p_x^2+p_y^2}} \qquad \dot{y} = \frac{p_y}{\sqrt{1+p_x^2+p_y^2}}$$

From Legendre eqns $\frac{d}{dt}p_x = \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$

$$\Rightarrow \dot{p}_x = -\frac{\partial V}{\partial x}, \quad \dot{p}_y = -\frac{\partial V}{\partial y}$$

Q1.(b) Show carefully that the following Hamiltonian is **integrable** and give it's integrals of motion.

$$H(\theta_1, \theta_2, I_1, I_2) = I_1^2 + I_2^2 + I_1 I_2 \cos(\theta_1 - 2\theta_2) + I_1^2 \cos(2\theta_1 - 4\theta_2) \qquad \text{(4 marks)}$$

Since $\left.\frac{\partial H}{\partial t}\right. = 0 \Rightarrow$ H is a constant. To find a second constant of the motion transform to new angle variables

$\phi_1 = \theta_1 - 2\theta_2$ and $\phi_2 = \theta_2$, via an F_2 generating function, where $F_2(\theta_1, \theta_2, J_1, J_2)$ and

$$\phi_i = \frac{\partial F_2}{\partial J_i}, \quad I_i = \frac{\partial F_2}{\partial \theta_i} \quad i = 1, 2$$

$$\Rightarrow \theta_1 - 2\theta_2 = \frac{\partial F_2}{\partial J_1}, \quad \theta_2 = \frac{\partial F_2}{\partial J_2} \Rightarrow \underline{F_2 = J_1(\theta_1 - 2\theta_2) + J_2\theta_2}$$

$$\Rightarrow I_1 = J_1 \text{ and } I_2 = -2J_1 + J_2$$

The Hamiltonian becomes

$$H(\cancel{\phi_i}\,\phi_i, J_i) = J_1^2 + (J_2 - 2J_1)^2 + J_1(J_2 - 2J_1)\cos\phi_1 + J_1^2 \cos 2\phi_1$$

Since H is independent of ϕ_2 ie $\frac{\partial H}{\partial \phi_2} = 0$

$$\Rightarrow \dot{J}_2 = \frac{-\partial H}{\partial \phi_2} = 0 \Rightarrow J_2 \text{ is a constant of the motion.}$$

$\Rightarrow$ Two constants of motion are

$\underline{H}$ and $\underline{J_2 = I_2 + 2I_1}$

$\Rightarrow$ integrable.

Question 2 on next page.

Q2.(a) Give the **primary resonance conditions** for the following Hamiltonian as a condition on the action I.

$$H(\theta, I, t) = 2I^2 + \epsilon\left(I\cos(2\theta - 2t) + \frac{I}{2}\cos(3\theta - 2t)\right)$$

(2 marks)

The resonance condition associated with a term of form $V(I)\cos(m\theta - nt)$ is $m\,\omega(I) - n = 0$,

where $\omega(I) = \dfrac{\partial H_0}{\partial I}$.

Here $\omega(I) = 4I$. $(H_0 = 2I^2)$

$\Rightarrow$ ~~reson~~ primary resenace conditions are

$2\omega(I) - 2 = 0 \Rightarrow 4I = 1 \Rightarrow I = 1/4$

and $3\omega(I) - 2 = 0 \Rightarrow 12I = 2 \Rightarrow I = 1/6$

Q2.(b) Assuming that ϵ is small, find the **width of the resonance** due to the $I\cos(2\theta-2t)$ term in the Hamiltonian from Q2.(a)

$$H(\theta, I, t) = 2I^2 + \epsilon\left(I\cos(2\theta-2t) + \frac{I}{2}\cos(3\theta-2t)\right),$$ to lowest order in ϵ.

To do this you will need to ignore the $\frac{I}{2}\cos(3\theta-2t)$ resonant term. (4 marks)

Ignoreing the $\frac{I}{2}\cos(3\theta-2t)$ term, let $\phi = \theta - t$ and use an F_2 generating function to transform to a rotating frame.

$$\phi = \theta - t, \quad \phi = \frac{\partial F_2}{\partial J} \Rightarrow F_2 = J(\theta - t)$$
$$\Rightarrow I = J$$

and the new Hamiltonian is $K = H + \frac{\partial F_2}{\partial t}$

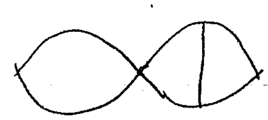
$$K = 2J^2 - J + \epsilon J \cos 2\phi$$

Since the resonance occurs at $J = \frac{1}{4}$

let $J = \frac{1}{4} + \Delta J$

$$\Rightarrow K = \left(2\left(\tfrac{1}{4}+\Delta J\right) - 1\right)\left(\tfrac{1}{4}+\Delta J\right) + \epsilon\left(\tfrac{1}{4}+\Delta J\right)\cos 2\phi$$
$$\simeq -\frac{1}{8} + 2\Delta J^2 + \frac{\epsilon}{4}\cos 2\phi + O(\epsilon \Delta J)$$

On the separatrix $K = -\frac{1}{8} + \frac{\epsilon}{4}$.
$(\Delta J = 0 \;\; \phi = 0)$



$\Rightarrow$ the width is found by taking $\phi = \frac{\pi}{2}$

$$-\frac{1}{8} + \frac{\epsilon}{4} = -\frac{1}{8} + 2\Delta J^2 - \frac{\epsilon}{4}$$

$$\Rightarrow \Delta J^2 = \frac{2\epsilon}{2\times 4} \Rightarrow \Delta J = \pm\frac{\sqrt{\epsilon}}{2}$$

and the width is $\underline{\sqrt{\epsilon}}$

Question 2 continued on next page.

Q2.(c) Find approximate **action angle** variables for the following Hamiltonian

$$H(\theta, I, t) = I^2 + \epsilon I \cos(3\theta + t) \qquad \text{for } \epsilon \text{ small,}$$

valid to order 1 in ϵ. (4 marks)

Transform to new action variable J

s.t., to order 1 in ϵ, $H(J)$ only.

To do this use a near identity generating function

$$F_2(\theta, J, t) = \theta J + \epsilon g(J) \sin(3\theta + t)$$

Then $I = \dfrac{\partial F_2}{\partial \theta} = J + 3\epsilon g(J) \cos(3\theta + t)$

and the new Hamiltonian

$$K = H(\theta, I(J, \theta), t) + \frac{\partial F_2}{\partial t}$$

$$= \left(J + 3\epsilon g(J) \cos(3\theta + t)\right)^2$$
$$+ \epsilon \left(J + 3\epsilon g(J) \cos(3\theta + t)\right) \cos(3\theta + t) + \epsilon g(J) \cos(3\theta + t)$$

To first order in ϵ this is

$$K = J^2 + 6\epsilon J g(J) \cos(3\theta + t)$$
$$+ \epsilon J \cos(3\theta + t) + \epsilon g(J) \cos(3\theta + t)$$

Eliminating the oscillating terms gives $K(J) = J^2$:

$$(6J g(J) + g(J)) + J = 0 \Rightarrow g(J) = \frac{-J}{(6J+1)}$$

$\Rightarrow$ Assuming $J \neq -1/6$ $\Rightarrow I = J - \dfrac{3\epsilon J}{6J+1} \cos(3\theta + t)$

and the new action is $\Rightarrow J = I + \dfrac{3\epsilon I}{6I+1} \cos(3\theta + t)$,

angle is $\phi = \dfrac{\partial F_2}{\partial \theta} = \theta + \dfrac{3\epsilon \sin(3\theta + t)}{(6I+1)^2}$ to first order in ϵ.

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Q3.(a) Give the continued fraction expansion of the positive root of $x^2 - 5x - 1 = 0$.

(3 marks)

$$x = \frac{5 \pm \sqrt{25+4}}{2} = \frac{5}{2} \pm \frac{\sqrt{29}}{2}$$

So $x_+ = \dfrac{5+\sqrt{29}}{2}$ and $5 < x_+ < 6$

$\Rightarrow$ in the continued fraction expansion

$a_0 = [x_+] = 5$

$$\Rightarrow \omega_1 = \frac{1}{x_+ - 5} = \frac{1}{\frac{\sqrt{29}-5}{2}} = \frac{2(\sqrt{29}+5)}{(\sqrt{29})^2 - 25}$$

$$= \frac{\sqrt{29}+5}{2} = x_+$$

Hence $a_n = 5 \quad \forall\, n = 0, 1 \ldots$

and the continued fraction expansion of x_+ is $[5\ 5 \ldots] = [\overline{5}]$

Question 3 continued on next page.

Q3. (b) Show that the following map evolves canonically

$$\theta_{n+1} = \theta_n + \frac{A}{J_{n+1}}$$

$$J_{n+1} = J_n + f(\theta_n)$$

by finding a generating function for the map. (3 marks)

Let the generating function be $F(\theta_n, J_{n+1})$ where

$\theta_{n+1} = \frac{\partial F}{\partial J_{n+1}}$ and $J_n = \frac{\partial F}{\partial \theta_n}$

$\Rightarrow \frac{\partial F}{\partial J_{n+1}} = \theta_n + \frac{A}{J_{n+1}}$ and $J_{n+1} - f(\theta_n) = \frac{\partial F}{\partial \theta_n}$

$\Rightarrow F = \theta_n J_{n+1} + A \ln(J_{n+1}) - \int f(\theta_n)\, d\theta_n$

The existence of such a function $\Rightarrow$ that the map evolves canonically.

Question 3 continued on next page.

Q3.(c) Show that the map

$$\theta_{n+1} = \theta_n + \tfrac{A}{J_{n+1}}$$
$$J_{n+1} = J_n + f(\theta_n)$$

is a product of two involutions, one of which is

$$I_1 \begin{pmatrix} \theta \\ J \end{pmatrix} = \begin{pmatrix} -\theta \\ J + f(\theta) \end{pmatrix} \quad \text{provided that } f(-\theta) = -f(\theta).$$

(4 marks)

The second involution is

$$I_2 \begin{pmatrix} \theta \\ J \end{pmatrix} = \begin{pmatrix} -\theta + \frac{A}{J} \\ J \end{pmatrix}$$

$$\Rightarrow I_2 I_1 \begin{pmatrix} \theta \\ J \end{pmatrix} = I_2 \begin{pmatrix} -\theta \\ J + f(\theta) \end{pmatrix} = \begin{pmatrix} -(-\theta) + \frac{A}{J+f(\theta)} \\ J + f(\theta) \end{pmatrix}$$

as above.

Both I_1 & I_2 are involutions ie I_i^2 is the identity:

$$I_1^2 \begin{pmatrix} \theta \\ J \end{pmatrix} = I_1 \begin{pmatrix} -\theta \\ J + f(\theta) \end{pmatrix} = \begin{pmatrix} -(-\theta) \\ J + f(-\theta) + f(-\theta) \end{pmatrix} = \begin{pmatrix} \theta \\ J \end{pmatrix}$$

since $f(\theta) = -f(\theta)$

$$I_2^2 \begin{pmatrix} \theta \\ J \end{pmatrix} = I_2 \begin{pmatrix} -\theta + A/J \\ J \end{pmatrix} = \begin{pmatrix} -(-\theta + A/J) + A/J \\ J \end{pmatrix} = \begin{pmatrix} \theta \\ J \end{pmatrix}$$