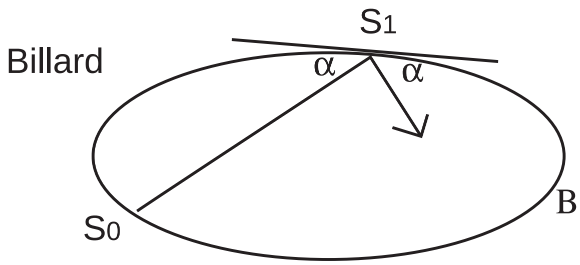
Section 1. Classical Dynamics

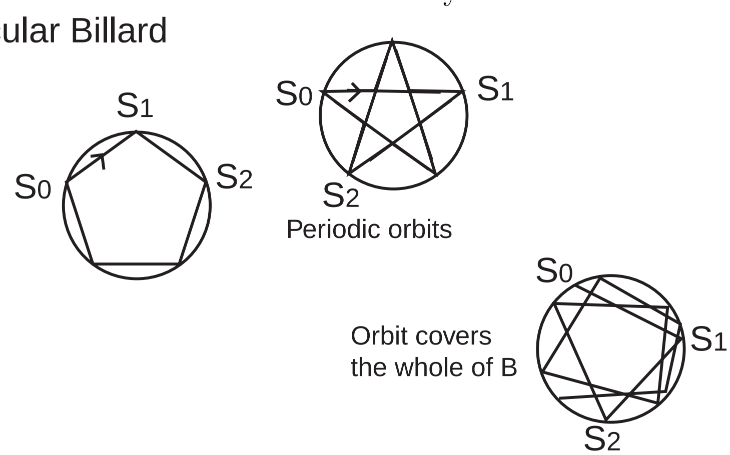
Section 1.10 Chaos in Billards.

Consider a particle moving freely in a region of the plane bounded by a closed curve B.

Assume that the particle moves without friction and is reflected elastically when it hits B. In this case the angle of reflection equals the angle of incidence.



Suppose as a simple example B is a circle. Then there are certain orbits which close. These orbits trace out regular ploygons such as pentagons or star shapes. But there are other orbits which never close so that they cover the whole boundary.



Any orbit can be specified by the arc position s_i on the boundary and the angle it's trajectory makes with the tangent to B at s. So one can think of the orbit as a map.

$$(s_0,\, \alpha_0) \longrightarrow (s_1,\, \alpha_1) \longrightarrow (s_2,\, \alpha_2)...$$

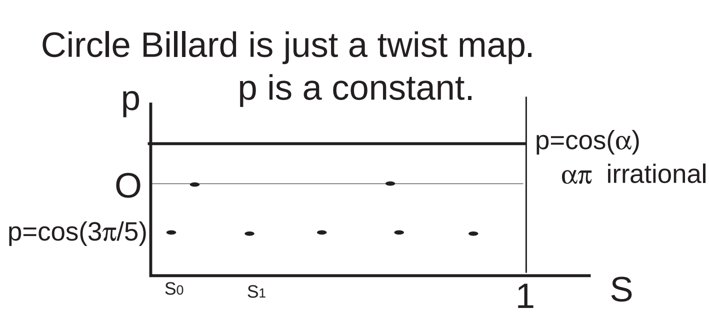
However it turns out to be more convienient to think interms of the tangential momentum

$$p = \cos\alpha \qquad \begin{pmatrix} s_{n+1} \\ p_{n+1} \end{pmatrix} = M \begin{pmatrix} s_n \\ p_n \end{pmatrix} \quad \text{on } S \times \mathbb{R}$$

where M depends on the boundary B.

If B is a circle $\alpha_0 = \alpha_1 = \alpha_i$, so that p is a constant of the motion.

If $\alpha = \frac{\pi K}{N}$ the orbit closes after N bounces. For instance if $\alpha = \pi \to p = 0$ it closes after two bounces. If $\alpha = \frac{3\pi}{5} \to p = \cos(\frac{3\pi}{5})$ closes after 5 bounces.



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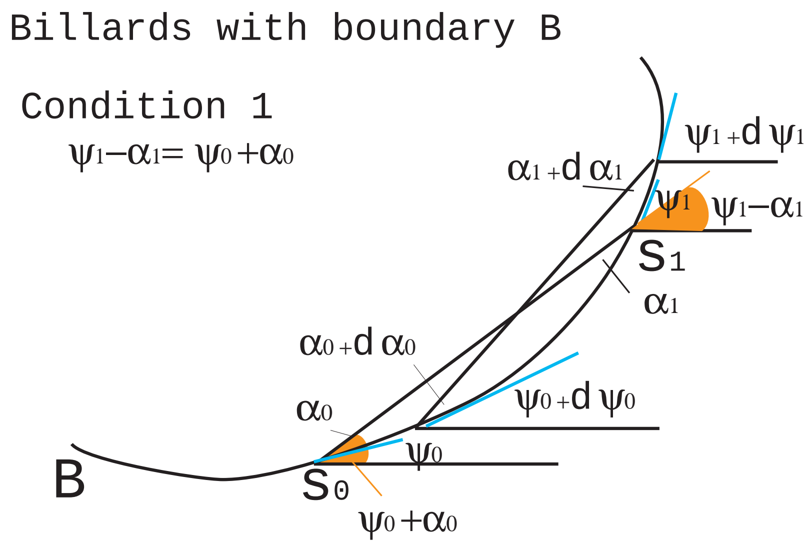
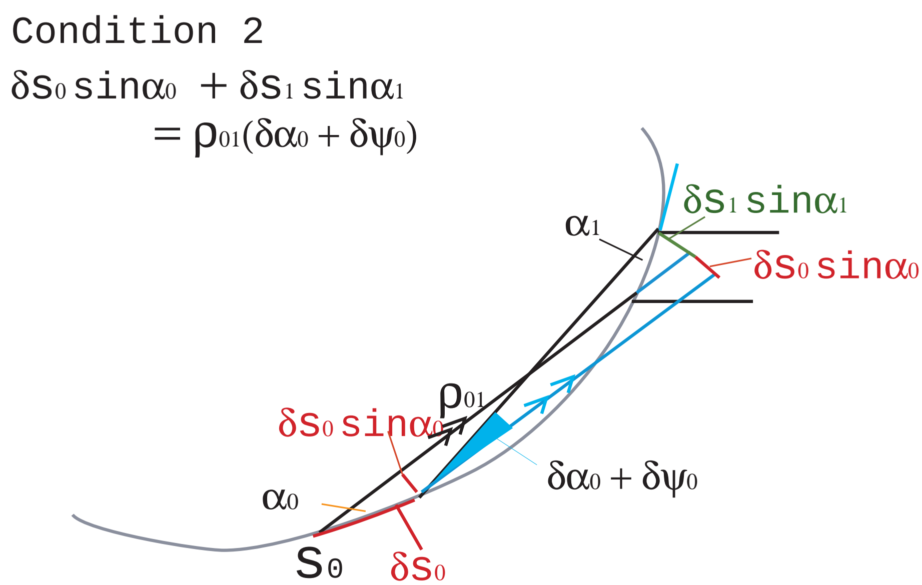
Any elastic billard mapping is area preserving

Proving this result is tricky. The canonical variables are $(s,\, p)$ as described. So what we need to prove is that

$$\frac{\partial(s_1,\, p_1)}{\partial(s_0,\, p_0)} = det \begin{pmatrix} \frac{\partial s_1}{\partial s_0} & \frac{\partial s_1}{\partial p_0} \\ \frac{\partial p_1}{\partial s_0} & \frac{\partial p_1}{\partial p_0} \end{pmatrix} = 1$$

Consider the small change $\alpha_0 \to \alpha_0 + d\alpha_0$. Define ψ as the angle the forward tangent to B at s_0 makes with the horiziontal, then ψ also undergoes a small change: $\psi_0 \to \psi_0 + d\psi_0$.





Then from the diagram $\psi_1 - \alpha_1 = \psi_0 + \alpha_0$ and if ds_0, ds_1, $d\alpha_1$, $d\psi_0$ are small

$$\delta s_0 \sin\alpha_0 + \delta s_1 \sin\alpha_1 = \rho_{01}(\delta\alpha_0 + \delta\psi_0)$$

Since $\psi_i(s_i)$ only and s_0 and α_0 are independent

$$\Rightarrow \frac{\partial s_1}{\partial s_0} = -\frac{\sin\alpha_0}{\sin\alpha_1} + \rho_{01}\frac{\partial\psi_0}{\partial s_0}$$

$$\text{and} \quad \frac{\partial s_1}{\partial\alpha_0} = \frac{\rho_{01}}{\sin\alpha_1}$$

Now the radius of curvature is

$$R(\psi) = \frac{ds}{d\psi} \Rightarrow s(\psi) = \int_{\frac{\pi}{2}}^{\psi} d\psi' R(\psi') \Rightarrow \frac{ds_0}{d\psi_0} = R(\psi_0) \quad \text{and} \quad \frac{ds_1}{d\psi_1} = R(\psi_1)$$

so that since s_0 and α_0 are independent ($\frac{\partial \alpha_0}{\partial s_0} = 0$)

$$\frac{\partial s_1}{\partial s_0} = -\frac{\sin\alpha_0}{\sin\alpha_1} + \frac{\rho_{10}}{\sin\alpha_1}\frac{d\psi_0}{ds_0} = -\frac{\sin\alpha_0}{\sin\alpha_1} + \frac{\rho_{01}}{R(\psi_0)\sin\alpha_1}$$

And since $p = \cos\alpha$

$$\frac{\partial s_1}{\partial p_0} = -\frac{1}{\sin\alpha_0}\frac{\partial s_1}{\partial \alpha_0} = -\frac{\rho_{10}}{\sin\alpha_0 \sin\alpha_1}$$

$$\frac{\partial p_1}{\partial s_0} = -\sin\alpha_1 \frac{\partial \alpha_1}{\partial s_0} = -\sin\alpha_1 \left(\frac{\partial \psi_1}{\partial s_0} + \frac{\partial \psi_0}{\partial s_0} \right)$$

But $\frac{\partial \psi_1}{\partial s_0} = \frac{\partial \psi_1}{\partial s_1}\frac{\partial s_1}{\partial s_0} = -\frac{\sin\alpha_0}{\sin\alpha_1} + \frac{\rho_{01}}{R(\psi_0)\sin\alpha_1}$ so that

$$\frac{\partial p_1}{\partial s_0} = \frac{\sin\alpha_0}{R(\psi_1)} + \frac{\sin\alpha_1}{R(\psi_0)} - \frac{\rho_{01}}{R(\psi_0)R(\psi_1)}$$

Lastly

$$\frac{\partial p_1}{\partial p_0} = \frac{\sin\alpha_1}{\sin\alpha_0}\frac{\partial \alpha_1}{\partial \alpha_0}$$

Where since ψ_0 is not a function of α_0 then $\frac{\partial \alpha_1}{\partial \alpha_0} = -1 + \frac{\partial \psi_1}{\partial s_1}\frac{\partial s_1}{\partial \alpha_1} = -1 + \frac{\rho_{01}}{R(\psi_1)\sin\alpha_1}$

$$\frac{\partial p_1}{\partial p_0} = \frac{\rho_{01}}{R(\psi_1)\sin\alpha_0} - \frac{\sin\alpha_1}{\sin\alpha_0}$$

This means that

$$\begin{pmatrix} \frac{\partial s_1}{\partial s_0} & \frac{\partial s_1}{\partial p_0} \\ \frac{\partial p_1}{\partial s_0} & \frac{\partial p_1}{\partial p_0} \end{pmatrix} = \begin{pmatrix} \left(-\frac{\sin\alpha_0}{\sin\alpha_1} + \frac{\rho_{01}}{R(\psi_0)\sin\alpha_1}\right) & -\frac{\rho_{01}}{\sin\alpha_0\sin\alpha_1} \\ \left(\frac{\sin\alpha_0}{R(\psi_1)} + \frac{\sin\alpha_1}{R(\psi_0} - \frac{\rho_{01}}{R(\psi_0)R(\psi_1)}\right) & \left(\frac{\rho_{01}}{\sin\alpha_0 R(\psi_1)} - \frac{\sin\alpha_1}{\sin\alpha_0}\right) \end{pmatrix}$$

The determinant is then

$$\frac{\partial s_1}{\partial s_0}\frac{\partial p_1}{\partial p_0} - \frac{\partial s_1}{\partial p_0}\frac{\partial p_1}{\partial s_0} = \left(-\frac{\sin\alpha_0}{\sin\alpha_1} + \frac{\rho_{01}}{R(\psi_0)\sin\alpha_1}\right)\left(\frac{\rho_{01}}{R(\psi_1)\sin\alpha_0} - \frac{\sin\alpha_1}{\sin\alpha_0}\right) +$$

$$\frac{\rho_{10}}{\sin\alpha_0\sin\alpha_1}\left(\frac{\sin\alpha_0}{R(\psi_1)} + \frac{\sin\alpha_1}{R(\psi_0} - \frac{\rho_{01}}{R(\psi_0)R(\psi_1)}\right) = 1$$

Calculating the Map

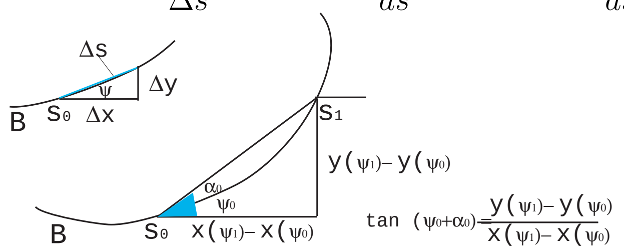
Given the curvature at any point on the boundary B you can obtain the map

$$\psi_1(\psi_0,\,\alpha_0),\,\alpha_1(\psi_0,\,\alpha_0)$$

numerically. From this you can deduce the map on $(s,\,p)$ through

$$s(\psi)=\int_{\frac{\pi}{2}}^{\psi}R(\psi')d\psi' \qquad p(\alpha)=\cos\alpha.$$

Consider the slope of the arc between s_0 and s_1. As s changes so does ψ. Let $(x,\,y)$ be a point on B. Then $\dfrac{\Delta x}{\Delta s}\approx\cos\psi\Rightarrow\dfrac{dx}{ds}=\cos\psi \qquad \dfrac{dy}{ds}=\sin\psi$



so that

$$x(\psi_1)-x(\psi_0)=\int\cos\psi ds=\int_{\psi_0}^{\psi_1}R(\psi)\cos\psi d\psi$$

Similarly $y(\psi_1)-y(\psi_0)=\displaystyle\int_{\psi_0}^{\psi_1}R(\psi)\sin\psi d\psi$ But the slope of the arc between s_0 and s_1 is $\psi_0+\alpha_0$

$$\frac{y(\psi_1)-y(\psi_0)}{x(\psi_1)-x(\psi_0)}=\tan(\psi_0+\alpha_0)$$

so that the equation

$$\tan(\psi_0+\alpha_0)=\frac{\int_{\psi_0}^{\psi_1}R(\psi)\sin\psi d\psi}{\int_{\psi_0}^{\psi_1}R(\psi)\cos\psi d\psi}$$

defines $\psi_1(\psi_0,\,\alpha_0)$. Then we can use the fact that $\psi_1-\alpha_1=\psi_0+\alpha_0$ to give

$$\alpha_1(\psi_0,\,\alpha_0)=\psi_1(\psi_0,\,\alpha_0)-\psi_0-\alpha_0$$

Integrability

Like all maps the map is integrable if there is a constant of the motion.

$$F(s_1,\,p_1)=F(s_0,\,p_0)$$

The simplest billard, the circular billiard is integrable because

$$\alpha_1=\alpha_0\Rightarrow p_1=p_0 \quad \text{so that } p \text{ is a constant.}$$

Also since, $\psi_1-\alpha_1=\psi_0+\alpha_0\Rightarrow\psi_1=\psi_0+2\alpha_0$.
So using the fact that the curvature is the radius R we have that

$$s(\psi)=\int_{\frac{\pi}{2}}^{\psi}Rd\psi'=R(\psi-\frac{\pi}{2})\Rightarrow \qquad s_1=s_0+2R\cos^{-1}p_0 \quad \text{and} \quad p_1=p_0$$

Fixed Points and Stability

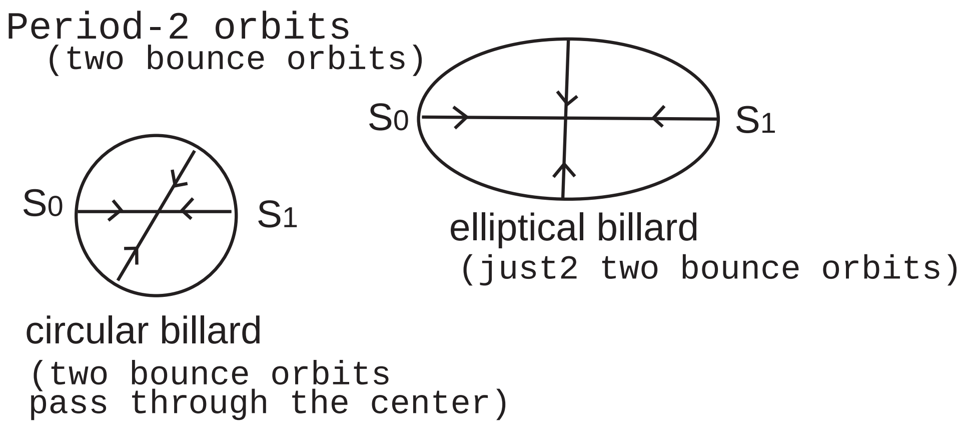
There are no period-1 orbits, but there are two bounce orbits which are often easy to identify. In the *elliptical billiard* they exist along the major and minor axes. To work out their stability we need to work out the linearized matrix. For any two bounce orbit $\alpha_0 = \alpha_1 = \frac{\pi}{2}$ and $R(\psi_0) = R(\psi_1) = R$so that

$$\frac{\partial(s_1,\, p_1)}{\partial(s_0,\, p_0)} = \left(\begin{array}{cc} \left(-\frac{\sin\alpha_0}{\sin\alpha_1} + \frac{\rho_{01}}{R(\psi_0)\sin\alpha_1}\right) & -\frac{\rho_{01}}{\sin\alpha_0\sin\alpha_1} \\ \left(\frac{\sin\alpha_0}{R(\psi_1)} + \frac{\sin\alpha_1}{R(\psi_0} - \frac{\rho_{01}}{R(\psi_0)R(\psi_1)}\right) & \left(\frac{\rho_{01}}{\sin\alpha_0 R(\psi_1)} - \frac{\sin\alpha_1}{\sin\alpha_0}\right) \end{array} \right) = \left(\begin{array}{cc} \frac{\rho}{R} - 1 & -\rho \\ \frac{2}{R} - \frac{\rho}{R^2} & \frac{\rho}{R} - 1 \end{array} \right)$$

$$\left(\frac{\partial(s_1,\, p_1)}{\partial(s_0,\, p_0)}\right)^2 = \left(\begin{array}{cc} 2\left(\frac{\rho}{R} - 1\right)^2 - 1 & 2\rho\left(1 - \frac{\rho}{R}\right) \\ \frac{2}{R}\left(\frac{\rho}{R} - 1\right)\left(2 - \frac{\rho}{R}\right) & \left(\frac{\rho}{R} - 1\right)^2 - 1 \end{array} \right)$$

This has trace $4\left(\frac{\rho}{R} - 1\right)^2 - 2$. So the orbit is stable if $-2 < 4\left(\frac{\rho}{R} - 1\right)^2 - 2 < 2$. That is

$$\left(\frac{\rho}{R} - 1\right)^2 < \quad \text{and if } \rho > R \quad \Rightarrow \quad \text{that the orbit is stable if } \frac{\rho}{R} < 2$$



Period-2 points for the Circular billiard.

Since for every period-2 orbit in a circular billiard $\rho = 2R$ so that $\frac{\rho}{R} = 2$. All two bounce orbits are neutrally stable. Infact for the circular billiard

$$Df = \left(\begin{array}{cc} \frac{\partial s_1}{\partial s_0} & \frac{\partial s_1}{\partial p_0} \\ \frac{\partial p_1}{\partial s_0} & \frac{\partial p_1}{\partial p_0} \end{array} \right) = \left(\begin{array}{cc} 1 & -\frac{2R}{\sin\alpha} \\ 0 & 1 \end{array} \right)$$

$$Df^n = \left(\begin{array}{cc} 1 & -\frac{2R}{\sin\alpha} \\ 0 & 1 \end{array} \right)^n = \left(\begin{array}{cc} 1 & -\frac{2Rn}{\sin\alpha} \\ 0 & 1 \end{array} \right)$$

which has trace 2 so all periodic orbits are neutrally stable.

Period-2 points for the Elliptical billard.

If we parametrise the ellipse by λ

$$x = a\cos\lambda\cos hM \qquad y = a\sin\lambda\sin hM$$

the ellipse is

$$\left(\frac{x}{a\cos hM}\right)^2 + \left(\frac{y}{a\sin hM}\right)^2 = 1$$

which has eccentricity $e = \frac{1}{\cos h^2 M}$.

Now the curvature is

$$R(\psi) = \frac{ds}{d\psi} = \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2}\frac{d\lambda}{d\psi}$$

Since

$$\frac{dy}{dx} = \tan\psi = \frac{\frac{dy}{d\lambda}}{\frac{dx}{d\lambda}} = -\tan hM \cot\lambda$$

That is

$$\tan\psi = -\tan hM \cot\lambda \Rightarrow \frac{d\psi}{d\lambda} = \frac{\tan hM}{\sin^2\lambda \sec^2\psi}$$

which you can then use to evaluate $R(\psi)$

Exercise show that

$$R(\psi) = \frac{a \sin hM \cos hM}{\left(\cos h^2 M \sin^2\psi + \sinh^2 M \cos^2\psi\right)^{\frac{3}{2}}}$$

Now consider the two bounces on the major and minor axes. $\psi = \frac{\pi}{2}$ or π. On the major axis

$$R(\frac{\pi}{2}) = \frac{a \sin hM}{\cos h^2 M} \Rightarrow \rho = 2a\cos hM \Rightarrow \frac{\rho}{R} = \frac{2\cos h^3 M}{\sin hM} > 2 \quad \text{unstable}$$

On the minor axis

$$R(\pi) = \frac{a\cos hM}{\sin h^2 M} \Rightarrow \rho = 2a \sin hM \Rightarrow \frac{\rho}{R} = \frac{2\sin h^3 M}{\cos hM} < 2 \quad \text{stable}$$

Infact the elliptical billiard is also integrable.

$$F(s,\, p) = \frac{p^2 - e^2\cos^2\psi(s)}{1 - e^2\cos^2\psi(s)} \quad \text{is a constant of the map.}$$

Proving this involves alot of algebra.

The stadium considtes of two semicircles joined by straight lines. In the semi circles $R(\psi) = R$, but for the straight lines $R = \infty$.

The two bounce orbit, lengthways is unstable as $\frac{\rho}{R} = \frac{2R+\eta}{r} > 2$. But the family of nonisolated two bounce orbits across the stadium are neutrally stable.

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Period-2 orbits for the stadium.
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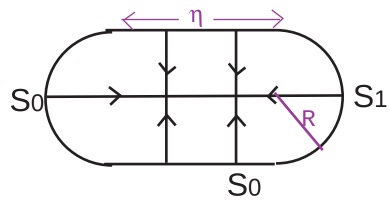

However the stadium is not integrable, infact there are no invariant lines and it has been shown that it is an ergodic billiard. That means that for almost every initial condition (s_0, p_0) the iterates will come arbitraily close to every point in the phase space as $n \to \infty$.

Oval billiards, not elliptical billiards provide us with a near integrable system. Let

$$R(\psi) = a(1 + \delta \cos 2\psi)$$

Then $\frac{ds}{d\psi} = R(\psi)$ and $\frac{dx}{ds} = \cos\psi$ implies that

$$\frac{dx}{d\psi} = a\cos\psi(1 + \delta\cos 2\psi) \Rightarrow x = a((1 + \frac{\delta}{2})\sin\psi + \frac{\delta}{6}\sin 3\psi$$

Similarily

$$y = a((-1 + \frac{\delta}{2})\cos\psi + \frac{\delta}{6}\cos 3\psi$$

and

$$s(\psi) = a(\psi - \frac{\pi}{2} + \frac{\delta}{2})\sin 2\psi$$

There two bounce orbits at $\psi = \frac{\pi}{2}, \frac{3\pi}{2}$ and at $\psi = 0\pi$. Like the ellipse the long one is unstable.

$$R(\frac{\pi}{2}) = a(1 - \delta) \quad \text{and} \quad \rho = 2a(1 + \frac{\delta}{3} \Rightarrow \frac{\rho}{R} = \frac{2(1 + \frac{\delta}{3})}{(1 - \delta)} > 2$$

Unlike the ellipse the oval is not integrable. As δ is increased from zero (the circular billiard) it appears to mimick the elliptical billiard but infact the separatrix is chaotic. Infact thought of as a perturbation from the elliptical billiard it exhibits all the usual resonance behaviour we expect from near integrable systems. The separatrix is chaotic and increasingly so as δ is increased from zero. Interesting four bounce orbits or period-4 orbits are seen for larger values of δ.