Section 1.2 Hamiltonian Mechanics

Lagranges equations can be formulated as a set of first order equations by using the momentum rather than the velocity. This elegant formulation, known as **Hamilton's equations** paves the way for future Hamiltonian-Jacobi theory, chaos theory and quantum mechanics.

Firstly define the **conjugate momentum** as

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad (1 \leq j \leq n)$$

Then from Lagranges equations of motion $\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$.

$$\dot{p}_j = \frac{\partial L}{\partial q_j} \qquad (1 \leq j \leq n)$$

However to transform completely to these new variables $(\mathbf{q},\ \mathbf{p},\ \mathbf{t})$, we must rewrite the Lagrangian in terms of the conjugate momenta, that is we must solve for $\dot{\mathbf{q}}(\mathbf{q},\ \mathbf{p},\ \mathbf{t})$.

Take the **orbit problem**. A particle of mass m is attracted by gravitation to a mass M fixed at the origin. In terms of polar coordinates the lagrangian is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{mMG}{r}.$$

The canonical momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

This means that

$$\dot{r} = \frac{p_r}{m} \quad \text{and} \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

Now Lagranges equations tell us that

$$\dot{p}_j = \frac{\partial L}{\partial q_j} \quad \Rightarrow \quad \dot{p}_r = \frac{\partial L}{\partial r} = mr\dot{\theta}^2 + \frac{mMG}{r^2}, \quad \text{and} \quad \dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0$$

So that Lagranges two equations of motion are equivalent to the following four first order equations.

$$\begin{aligned} \dot{r} &= \frac{p_r}{m}, & \dot{p}_r &= \frac{p_\theta^2}{mr^3} - \frac{mMG}{r^2} \\ \dot{\theta} &= \frac{p_\theta}{mr^2}, & \dot{p}_\theta &= 0 \end{aligned}$$

This is the **Hamiltonian form** of Lagranges equations.

Legendre Transformations

One might naively think that there are cases where one cannot derive the Hamiltonian form from the Lagrangian one. However the conversion from one form to the other is an example of a **Legendre Transformation** and can always be performed.

Consider a function $F(u_1, u_2)$ and define

$$v_1 = \frac{\partial F}{\partial u_1}, \qquad v_2 = \frac{\partial F}{\partial u_2}$$

Is it possible to write the inverse formulas in the form

$$u_1 = \frac{\partial G}{\partial v_1}, \qquad u_2 = \frac{\partial G}{\partial v_2}$$

for some function $G(v_1, v_2)$?

For a simple case, say $F = 2u_1^2 + 3u_1u_2 + u_2^2$

$$\Rightarrow \qquad v_1 = 4u_1 + 3u_2, \qquad v_2 = 3u_1 + 2u_2$$

Solving for u_1 and u_2 gives

$$u_1 = -2v_1 + 3v_2, \qquad u_2 = 3v_1 - 4v_2$$

So that $G = -v_1^2 + 3v_1v_2 - 2v_2^2$, works! But will it always? In fact it will.

Consider the expression

$$X = F(u_1, u_2) + G(v_1, v_2) - (u_1v_1 + u_2v_2),$$

If we think of $v_1(u_1, u_2)$ and $v_2(u_1, u_2)$ then $X(u_1, u_2)$. So that

$$\frac{\partial X}{\partial u_1} = \frac{\partial F}{\partial u_1} + \left(\frac{\partial G}{\partial v_1}\frac{\partial v_1}{\partial u_1} + \frac{\partial G}{\partial v_2}\frac{\partial v_2}{\partial u_1}\right) - \left(v_1 + u_1\frac{\partial v_1}{\partial u_1} + u_2\frac{\partial v_2}{\partial u_1}\right)$$

$$\frac{\partial X}{\partial u_1} = \left(\frac{\partial F}{\partial u_1} - v_1\right) + \left(\frac{\partial G}{\partial v_1} - u_1\right)\frac{\partial v_1}{\partial u_1} + \left(\frac{\partial G}{\partial v_2} - u_2\right)\frac{\partial v_2}{\partial u_1} = 0 + 0 + 0 = 0$$

Similarly $X(u_1, u_2)$ is independent of u_2. It follows that $X(u_1, u_2)$ is a constant. Without loss of generality we can take that constant to be zero as it could be absorbed into G. So that

$$F(u_1, u_2) + G(v_1, v_2) = (u_1v_1 + u_2v_2) \qquad \text{defines } G(v_1, v_2).$$

$$G(v_1, v_2) = (u_1v_1 + u_2v_2) - F(u_1, u_2).$$

(In the previous example
$G(v_1, v_2) = (-2v_1 + 3v_2)v_1 + (3v_1 - 4v_2)v_2 - 2(-2v_1 + 3v_2)^2 - 3(-2v_1 + 3v_2)(3v_1 - 4v_2) - (3v_1 - 4v_2)^2 = -v_1^2 + 3v_1v_2 - 2v_2^2$)

Active and Passive variables

The variables $\mathbf{u} = (u_1,\, u_2)$ and $\mathbf{v} = (v_1,\, v_2)$ are **active variables**, as they are transformed. However the functions F and G may contain **passive variables** that are not transformed. Suppose for instance that $F = F(u_1,\, u_2,\, w)$ then by definition G may also be a function of w since

$$G(v_1,\, v_2,\, w) = (u_1 v_1 + u_2 v_2) - F(u_1,\, u_2,\, w).$$

If v_1 and v_2 are defined as before

$$v_1 = \frac{\partial F}{\partial u_1}, \qquad v_2 = \frac{\partial F}{\partial u_2}$$

they may also be functions of w. However one can show that

$$\frac{\partial F}{\partial w} + \frac{\partial G}{\partial w} = 0 \qquad \text{for any passive variable.}$$

Legendre Transforms can be defined for any number of active and passive variables. If $\mathbf{u} = (u_1,\, u_2,\, ...u_n)$ are active variables and $\mathbf{w} = (w_1,\, ...w_n)$ are passive variables of $F(\mathbf{u},\, \mathbf{w})$ and $\mathbf{v}$ are functions of the active variables such that

$$\mathbf{v} = grad_{\mathbf{u}} F(\mathbf{u},\, \mathbf{w})$$

then the inverse formula can always be written in the form

$$\mathbf{u} = grad_{\mathbf{v}} G(\mathbf{v},\, \mathbf{w})$$

where the function $G(\mathbf{v},\, \mathbf{w})$ is related to the function $F(\mathbf{u},\, \mathbf{w})$ by the formula

$$G(\mathbf{v},\, \mathbf{w}) + F(\mathbf{u},\, \mathbf{w}) = \mathbf{u} \cdot \mathbf{v}$$

where $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$.

Also the derivatives of the F and G with respect to the passive variables are given by

$$grad_{\mathbf{w}} F(\mathbf{u},\, \mathbf{w}) = -grad_{\mathbf{w}} G(\mathbf{v},\, \mathbf{w})$$

The Hamiltonian

Hamilton's Equations of motion can be derived from the **Hamiltonian** of a system. Since they are a function of the conjugate momenta, which are defined by

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad (1 \leq j \leq n) \quad \text{or} \quad \mathbf{p} = grad_{\dot{\mathbf{q}}} L(\mathbf{q},\, \dot{\mathbf{q}},\, \mathbf{t})$$

rather than $\dot{\mathbf{q}}$, the first step is to eliminate the coordinate velocities $\dot{\mathbf{q}}$ in the Lagrangian in favor of the the momenta. This means that the formula for $\mathbf{p}$ must be inverted to express $\dot{\mathbf{q}}$ in terms of $\mathbf{q}$, $\mathbf{p}$, and t. This is what Legendre transforms do! In fact

$$\dot{\mathbf{q}} = grad_{\mathbf{p}} H(\mathbf{q},\, \mathbf{p},\, \mathbf{t}) \qquad \text{or} \quad \dot{q}_j = \frac{\partial H}{\partial p_j}$$

where the **Hamiltonian** H is the **Legendre transform** of the **Lagrangian** $L(\mathbf{q},\, \dot{\mathbf{q}},\, \mathbf{t})$.

This means that the **Hamiltonian** is given by

$$H(\mathbf{q},\,\mathbf{p},\,\mathbf{t}) = \dot{\mathbf{q}}\cdot\mathbf{p} - L(\mathbf{q},\,\dot{\mathbf{q}},\,\mathbf{t}),$$

or in component form

$$H = \sum_{j=1}^{n} \dot{q}_j p_j - L(\mathbf{q},\,\dot{\mathbf{q}},\,\mathbf{t}).$$

Since both time and $\mathbf{q}$ are passive variables in the Legendre transform

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

$$grad_{\mathbf{q}} L(\mathbf{q},\,\dot{\mathbf{q}},\,\mathbf{t}) = -grad_{\mathbf{q}} H(\mathbf{q},\,\mathbf{p},\,\mathbf{t})$$

Using this and Lagranges equations of motion gives $\dot{p}$ in terms of the Hamiltonian,

$$\dot{p}_j = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j} \qquad \text{in component form.}$$

So **Hamilton's Equations of motion** are

$$\dot{\mathbf{q}} = grad_{\mathbf{p}} H(\mathbf{q},\,\mathbf{p},\,\mathbf{t}) \qquad \dot{\mathbf{p}} = -grad_{\mathbf{q}} H(\mathbf{q},\,\mathbf{p},\,\mathbf{t})$$

or

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

Example For the orbit problem

$$H(r,\,\theta,\,p_r,\,p_\theta) = \dot{r}p_r + \dot{\theta}p_\theta - \left(\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{mMG}{r}\right)$$

Using the fact that

$$\dot{r} = \frac{p_r}{m} \qquad \text{and} \qquad \dot{\theta} = \frac{p_\theta}{mr^2},$$

$$H(r,\,\theta,\,p_r,\,p_\theta) = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{1}{2}m\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2\right) + \frac{mMG}{r} = \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{mMG}{r}$$

So Hamilton's equations of motion are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \quad \text{and} \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2},$$

and

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{mMG}{r^2} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

Not really an improvement if you want a practical solution!

Properties of the Hamiltonian and other notes.

- **1** If the Hamiltonian is not explicitly a function of time then it is a constant of the motion.
- **2** If the Hamiltonian is not explicitly a function of a generalized coordinate, say q_j, then the generalized momentum p_j is a constant of the motion.
- **3 Liouvilles's Theorem** The Phase space volume is preserved by the phase flow.
- **4 Poisson Brackets** Hamilton's equations have a particularly elegant form when written in terms of the Poisson Bracket.

1. If the Hamiltonian is not explicitly a function of time then it is a constant of the motion.

$$\frac{dH}{dt} = \sum_{j=1}^{n}\left(\frac{\partial H}{\partial q_j}\dot{q}_j + \frac{\partial H}{\partial p_j}\dot{p}_j\right) = \sum_{j=1}^{n}(-\dot{p}_j\dot{q}_j + \dot{q}_j\dot{p}_j) = 0$$

Systems where $H = H(\mathbf{q}, \mathbf{p})$ are called **autonomous**. For instance the Hamiltonian in the orbit problem is autonomous so $H = \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{mMG}{r}$ is a constant of the motion.

In fact for an autonomous Hamiltonian each phase path must lie on a "surface" of constant energy within the phase space. This means that a one degree of freedom, autonomous Hamiltonian system can be solved by quadrature.

Since

$$H(q, p) = H_0 \quad \text{we can solve for } p \text{ as a function of } q \quad p = f(q)$$

Now Hamilton's equation

$$\dot{q} = \frac{\partial H}{\partial p}(q, f(q)) \quad \text{is a function of } q,$$

so it can be separated and, in theory solved.

Take the example of the linear pendulum

$$H = \frac{p^2}{2m} + m\omega_0^2\frac{q^2}{2} = H_0 \quad \text{where } \omega_0 = \sqrt{\tfrac{g}{\ell}}$$

Then solving for p gives $p = \sqrt{\left(H_0 - m\omega_0^2\frac{q^2}{2}\right)2m.}$

From Hamilton's first equation $\dot{q} = \frac{p}{m} = \sqrt{2\left(H_0 - m\omega_0^2\frac{q^2}{2}\right)/m}$ so separating

$$\int \frac{dq}{\sqrt{\left(\frac{2H_0}{m\omega_0^2} - q^2\right)}} = \int \omega_0 dt$$

Solving for q gives

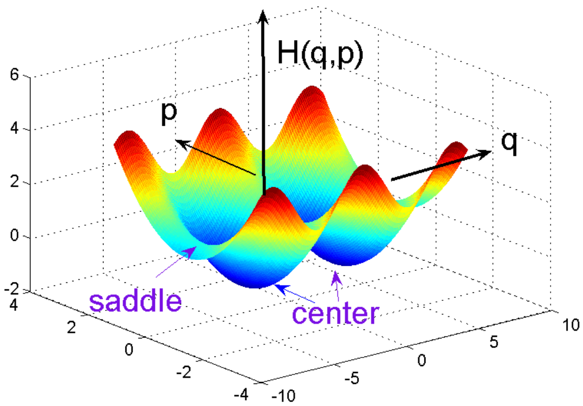
$$\Rightarrow q = \sqrt{\frac{2H_0}{m\omega_0^2}}\cos(\omega_0(t - t_0)) \quad p = -\sqrt{2H_0 m}\sin(\omega_0(t - t_0))$$

Although solutions are obviously useful the constant Hamiltonian can tell us a lot about the system.

Take the nonlinear Pendulum

$$H = \frac{p^2}{2m} - mg\ell\cos\frac{q}{\ell}$$

The curves of constant Energy, the contours of $H = H_0 = \frac{p^2}{2m} - mg\ell\cos\frac{q}{\ell}$, are the phase curves of the system. Thought of as a surface, the peaks and pits of the "Hamiltonian range" are the centers of the system and the mountain passes are the saddles.



The separatrices, given by $H = mg\ell$, connecting the saddles divide the periodic motion from the librational motion.

The nonlinear Pendulum can also be solved by quadrature, but the solutions are Elliptic integrals. Taking the simple case where $m = \ell = 1$ and $\omega_0^2 = g$,

$$H = \frac{p^2}{2} - \omega_0^2\cos q \qquad \Rightarrow \qquad p = \sqrt{2\left(H + \omega_0^2\cos q\right)}$$

So Hamilton's first equation is

$$\dot{q} = p = \sqrt{2\left(H + \omega_0^2\cos q\right)}$$

which can be separated and inside the separatrix has the solutions:

$$q = 2\arcsin(ksn(\omega_0(t-t_0),\, k)), \qquad p = 2\omega_0 kcn(\omega_0(t-t_0),\, k)$$

where $k = \sqrt{\dfrac{\omega_0^2 + H}{2\omega_0^2}} < 1$, since $H < \omega_0^2$. Actually on the separatrix, where $H = \omega_0^2$,

$$q = \arctan\left(e^{\omega_0(t-t_0)}\right) - \pi \qquad p = \frac{2\omega_0}{\cosh(\omega_0(t-t_0))}$$

Note that as $t \to \infty$ then $p \to 0$.

2. If the Hamiltonian is not explicitly a function of a generalized coordinate, say q_j, then the generalized momentum p_j is a constant of the motion.
This follows from the Hamilton equation

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}$$

For instance in the orbit problem the Hamiltonian $H = \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{mMG}{r}$ is not a function of θ, so that $\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$, which implies that p_θ is a constant.

Phase Space Reduction If the Hamiltonian is not explicitly a function of q_j, which means that p_j is a constant of the motion, then the dimension of the phase space is effectively reduced by two (the number of degrees of freedom is reduced by one). This will mean that a 2 degree of freedom system can be solved by quadrature. For instance, with p_θ as a constant in the orbit problem the Hamiltonian is effectively only a function of (r, p_r). So that

$$p_r = \sqrt{2m\left(H - \frac{1}{2}\frac{p_\theta^2}{mr^2}\right)} \quad \Rightarrow \dot{r} = \frac{p_r}{m} = \frac{1}{m}\sqrt{2m\left(H - \frac{1}{2}\frac{p_\theta^2}{mr^2}\right)}$$

which is separable for p_θ equal to a constant.

Note that **the Phase Space description is not unique.**
Because different Lagrangians can be used to describe the same physical system, even if the position variables are the same the momenta may differ.
Two Lagrangians for the same system may differ by a total time derivative.

$$L' = L + \frac{dF}{dt} \quad \Rightarrow p_j = \frac{\partial L}{\partial \dot{q}_j} \quad \text{but} \quad p'_j = \frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial}{\partial \dot{q}_j}\frac{dF}{dt}$$

Take the example of the driven pendulum.

$$L = \frac{1}{2}m\left(\ell^2\cos^2\theta\dot{\theta}^2 + \left(\dot{y}_s(t) + \ell\sin\theta\dot{\theta}\right)^2\right) - mg(y_s - \ell\cos\theta) \quad \Rightarrow \quad p = m\ell^2\dot{\theta} + m\dot{y}_s\ell\sin\theta$$

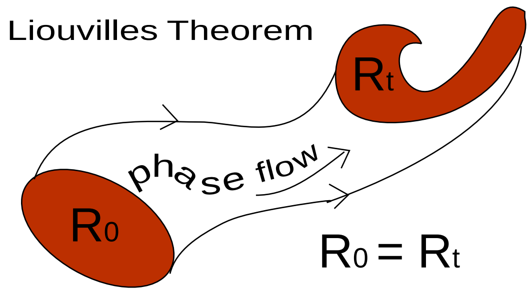
But

$$L' = \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\left(g + \ddot{y}_s(t)\right)\cos\theta \quad \Rightarrow \quad p' = m\ell^2\dot{\theta}.$$

This means that the phase space descriptions will be different.

3. Liouvilles Theorem

Liouville's Theorem says that the phase space volume is preserved by the phase flow. So, as time evolves, a region in the phase space may be stretched and sheared by the flow but its area will be preserved.



In the Proof we assume that the equations of motion are

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\, t) \quad \text{where } F_i \text{ have the particular form of Hamiltons equations of motion}$$

So we take $\mathbf{x} = (q_1,\, p_1,\, q_2,\, p_2 ...)$ this means that $F_1 = \frac{\partial H}{\partial p_1},\ F_2 = -\frac{\partial H}{\partial q_1},\ F_3 = \frac{\partial H}{\partial p_2}, \ldots$

For simplicity I will just take a one degree of freedom system, but the more general case is very similar.

Consider a set of phase points moving about the $(x_1,\, x_2)$ plane, which at time $t = 0$ occupies a region R_0. After a time t, a typical point $\mathbf{x}$ of R_0 has moved to a position $\mathbf{X} = \mathbf{X}(\mathbf{x},\, t)$ and the set as a whole now occupies the region R_t. (In two dimensions R_0 and R_t are areas in the $(x_1,\, x_2)$ plane, but in general they will be volumes.) This area is given by

$$V(t) = \int_{R_t} dX_1 dX_2 = \int_{R_0} J dx_1 dx_2 \quad \text{where} \quad J = det \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix}$$

is the Jacobian of the transformation $\mathbf{X} = \mathbf{X}(\mathbf{x},\, t)$.

Now for small t, $\mathbf{X}$ may be approximated by

$$\mathbf{X}(\mathbf{x},\, t) = \mathbf{X}(\mathbf{x},\, 0) + t\frac{\partial \mathbf{X}}{\partial t}(\mathbf{x},\, 0) + 0(t^2) = \mathbf{x} + t\mathbf{F}(\mathbf{x},\, 0) + 0(t^2)$$

This means that

$$J = 1 + t\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right)_{t=0} + 0(t^2) = 1 + t \ \text{div} \ \mathbf{F}(\mathbf{x},\, 0) + 0(t^2)$$

Substituted back into the equation for the area

$$V(t) = \int_{R_0} (1 + t \ \text{div} \ \mathbf{F}(\mathbf{x},\, 0))\, dx_1 dx_2 + 0(t^2)$$

where t is small, it follows that

$$\frac{dV(t)}{dt}|_{t=0} = \lim_{t\to 0}\left(\frac{V(t) - V(0)}{t}\right) = \int_{R_0} \text{div} \ \mathbf{F}(\mathbf{x},\, 0) dx_1 dx_2$$

However the value $t = 0$ was arbitrarily chosen and the result will hold for any t:

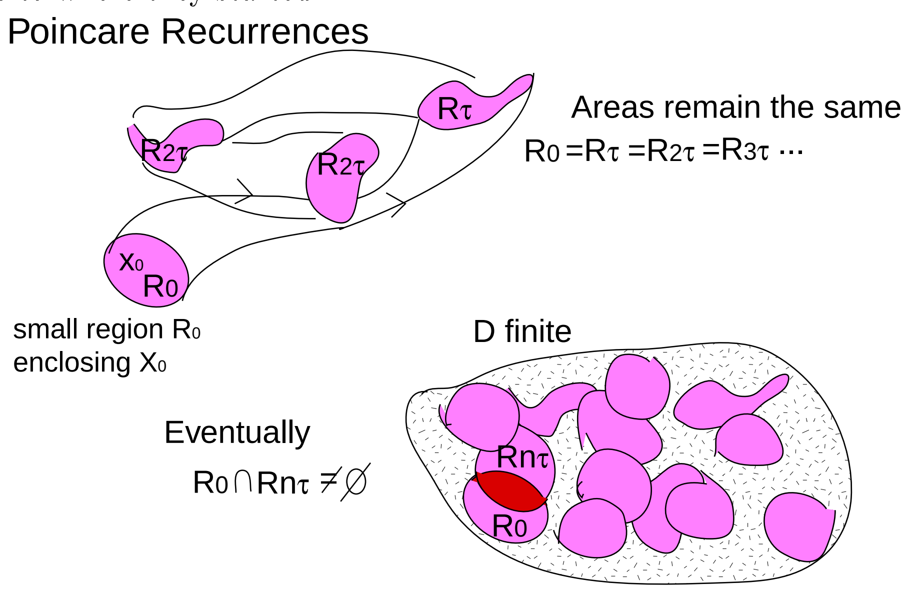
$$\frac{dV(t)}{dt} = \int_{R_t} \text{div } \mathbf{F}(\mathbf{x}, t) dx_1 dx_2$$

Now comes the point where we use the fact that the system is Hamiltonian. Recall that $\mathbf{x} = (q_1, p_1, q_2, p_2 ...)$ this means that $F_1 = \frac{\partial H}{\partial p_1}$, $F_2 = -\frac{\partial H}{\partial q_1}$, $F_3 = \frac{\partial H}{\partial p_2}$, ... So that

$$\text{div } \mathbf{F} = \frac{\partial}{\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial}{\partial p_1}\left(-\frac{\partial H}{\partial q_1}\right) + \cdots = 0 + 0 + \cdots = 0$$

So $\dfrac{dV(t)}{dt} = 0$, which proves the result. This means that there can be no attracting sets, such as stable periodic orbits.

Poincare Recurrence follows from Liouville's theorem. It only applies to autonomous Hamiltonian systems which are confined to some bounded region D. If this is the case then the theorem states that almost all trajectories eventually return arbitrarily close to where they started.



To prove this we must show that for any finite region R_0 there are points in the region that return to R_0. So take any finite neighborhood R_0 of $\mathbf{x_0}$ and consider successive images of R_0, which we shall call R_t, under time evolution. Consider the regions R_τ, $R_{2\tau}$, $R_{3\tau}$, ...$R_{n\tau}$. By Liouville's theorem all of these regions have the same volume. But we also know that all of these regions lie within some fixed volume. So as we increase n one of them must overlap a previous region.

Now suppose that it is $R_{m\tau}$ that overlaps with $R_{k\tau}$, for some $0 \le k \le m$. This means that there must be points $\mathbf{x_1} \in \mathbf{R_0}$ and $\mathbf{x_2} \in \mathbf{R_0}$ such that $\mathbf{X}(\mathbf{x_1}, m\tau) = \mathbf{X}(\mathbf{x_2}, k\tau)$.

Now we use the fact that the system is autonomous. In this case

$$\mathbf{X}(\mathbf{x_1}, m\tau) = \mathbf{X}(\mathbf{x_2}, k\tau) \quad \Rightarrow \quad \mathbf{X}(\mathbf{x_1}, (m-k)\tau) = \mathbf{X}(\mathbf{x_2}, 0) = \mathbf{x_2}$$

So the $\mathbf{x_1} \in \mathbf{R_0}$ evolves to $\mathbf{x_2}$ after a time $(m-k)\tau$, which is also in R_0. Since this phase point has returned to R_0 the theorem is proved.

The result is surprising because it implies that for any choice of initial conditions the system comes arbitrarily close to reassuming those conditions at a later time. Ofcourse the actual time frame is not specified.

4. The Poisson Bracket of two functions F and G is given by

$$\{F,\, G\} = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right)$$

If we now consider the evolution of any function $F(\mathbf{q},\, \dot{\mathbf{q}},\, t)$ under time

$$\frac{dF}{dt} = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial p_j}\dot{p}_j \right) + \frac{\partial F}{\partial t} = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial H}{\partial q_j} \right) + \frac{\partial F}{\partial t}$$

using Hamilton's equations of motion: $\dot{q}_j = \frac{\partial H}{\partial p_j}$ and $\dot{p}_j = -\frac{\partial H}{\partial q_j}$. This becomes

$$\frac{dF}{dt} = \{F,\, H\} + \frac{\partial F}{\partial t}$$

$$\text{Hamilton's equations} \quad \Rightarrow \quad \frac{dq_j}{dt} = \{q_j,\, H\}, \qquad \frac{dp_j}{dt} = \{p_j,\, H\}$$

and if the Hamiltonian is not a function of time $\dfrac{dH}{dt} = \{H,\, H\} = 0$.

Poisson brackets are antisymmetric, bilinear and satisfy Jacobi's identity.

$$\text{antisymmetric} \qquad \{F,\, G\} = -\{G,\, F\}$$

$$\text{bilinear} \quad \{F,\, G+H\} = \{F,\, G\} + \{F,\, H\}, \qquad \{F,\, cG\} = \{cF,\, G\} = c\{F,\, G\}$$

$$\text{Jacobi's identity} \qquad \{F,\, \{G,\, H\}\} + \{H,\, \{F,\, G\}\} + \{G,\, \{H,\, F\}\} = 0$$

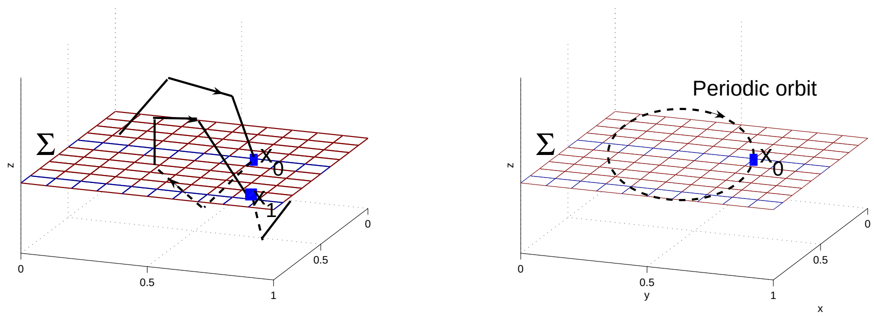
Also if F is time independent and conserved by the evolution under the Hamiltonian the Poisson Bracket of F with H is zero.

$$\text{If} \quad \frac{dF}{dt} = 0 \quad \text{and} \quad \frac{\partial F}{\partial t} = 0 \quad \Rightarrow \{F,\, H\} = 0$$

Periodically Driven Systems

The phase space of a time dependent one degree of freedom system is three dimensional or has $1\frac{1}{2}$ degrees of freedom. The Hamiltonian is no longer a constant of the motion and such systems cannot be solved by quadrature. In fact nonlinear $1\frac{1}{2}$ degree of freedom systems can have chaotic solutions. Their motion can be hard to visualize in 3D, so they are usually analyzed by considering the Poincaré Map.

A Poincaré section or map is a device invented by Henri Poincaré for analyzing systems of higher than two dimensions. Poincaré realized that much of the important information about a trajectory was encoded in the points in which the trajectory passed through a 2-dimensional plane, or surface of section Σ. The order of these intersection points defines a map in the surface of section Σ.



Each time the trajectory pierces Σ in a downward direction we record the point. The successive piercings are the successive iterates of the map. For instance if the trajectory is a simple periodic orbit the successive piercings give just one point and the point is a fixed or critical point of the map.

The simplest Poincaré maps to calculate are those of periodically forced systems. Consider the driven linear pendulum

$$H(q,\, p,\, t) = \frac{p^2}{2} + \omega_0^2 \frac{q^2}{2} - \epsilon q \cos(\omega t)$$

Hamilton's equations of motion are

$$\dot{q} = p \qquad \dot{p} = -\omega_0^2 q + \epsilon \cos(\omega t)$$

Or $\ddot{q} + \omega_0^2 q = \epsilon \cos(\omega t)$. This has the solution

$$q = A\cos(\omega_0(t - t_0)) + \frac{\epsilon}{\omega_0^2 - \omega^2}\cos(\omega t) \qquad \Rightarrow \text{quasi periodic motion for } \omega \neq \omega_0.$$

So solutions lie on a torus. (The phase space is $\mathbb{R}^2 \times S$.) This is most easily thought of as a surface in 3D with the plane $t = \frac{2\pi}{\omega}$ identified with the plane $t = 0$. The Poincaré maps is then a stroboscopic map $(q_n,\, p_n) = (q(\frac{2n\pi}{\omega} + t_0),\, p(\frac{2n\pi}{\omega} + t_0))$.

If $\frac{\omega_0}{\omega} = \frac{n}{m}$, i.e. a rational, the motion is periodic. If $\frac{\omega_0}{\omega}$ is irrational the trajectory covers the torus, passing arbitrarily close to any point on the torus.

The Poincaré map for the case $\frac{\omega_0}{\omega} = \frac{n}{m}$ consists of m points or iterates. The solution is said to be a period-m orbit, it is a periodic orbit. If $\frac{\omega_0}{\omega}$ is irrational the Poincaré map fills out a closed invariant curve. Such behavior is indicative of regular (quasi periodic rather than chaotic) trajectories and typical of linear forced systems.

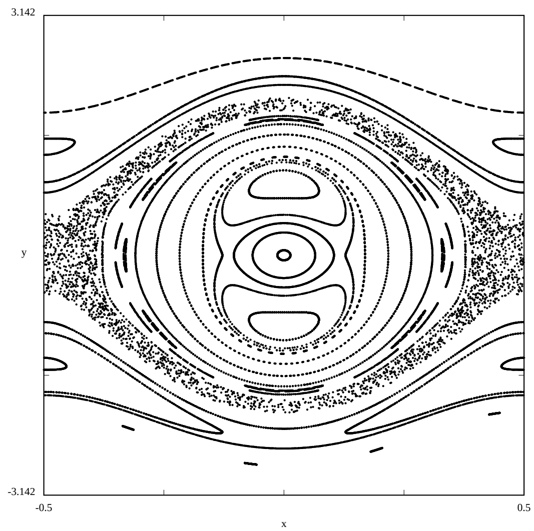
Nonlinear systems are typically quite different. In fact some level of chaos is often present. (We will cover how to tell if a system has chaotic solutions later.) For the moment we will take two, rather different cases of systems with chaos and look at what their Poincaré maps tell us.

The first system is a $1\frac{1}{2}$ degree of freedom system; the **Driven Nonlinear Pendulum**

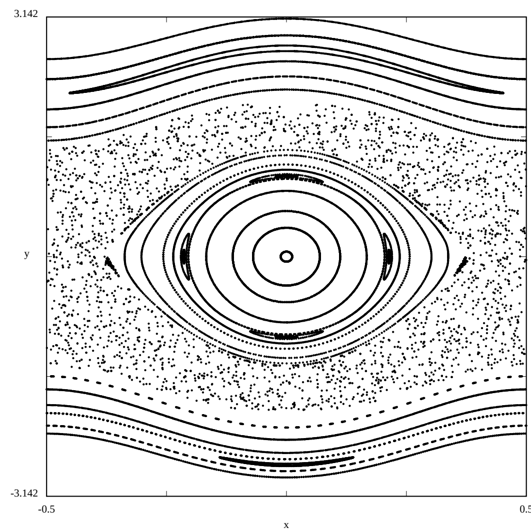
$$H(q,\, p,\, t) = \frac{p^2}{2} - \omega_0^2(1 + \epsilon\cos(\omega t))\cos q$$

If $\epsilon = 0$ this is the nonlinear pendulum which can be solved by quadrature. However even for small ϵ this system has complex solutions. In the following Poincaré maps some initial conditions give iterates that trace out what appear to be simple closed curves. This suggests the presence of tori and trajectories which eventually cover them. Other trajectories, possibly chaotic, exist, particularly where the separatrix would be for the case with $\epsilon = 0$.

$\omega_0 = 1$, $\omega = 1.9$ and $\varepsilon = 0.025$

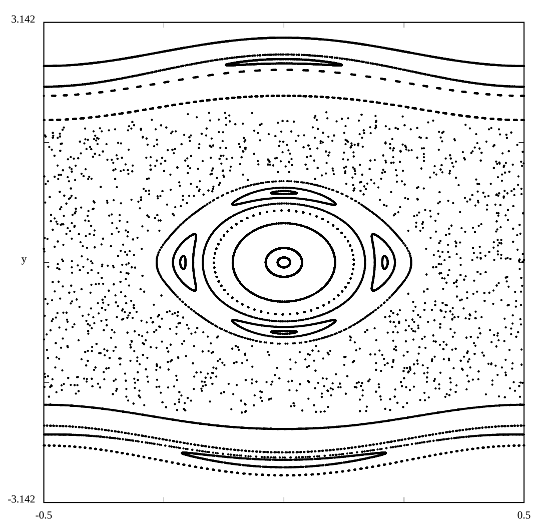


$\omega = 1.2$, $\varepsilon = 0.1$

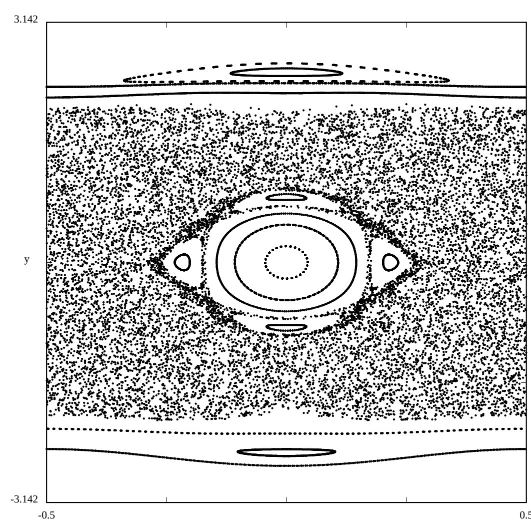


Also noticeable are resonance islands replacing the tori. The width of these islands increases with ϵ and their presence is in some sense the reason for the chaos. For much larger values of chaos the phase space is dominated by a chaotic sea, relieved only by the occasional resonance island.

$\omega = 1.2$, $\varepsilon = 0.2$



$\omega = 1.175$, $\varepsilon = 0.3$



Henon Hiles

The Henon Hiles system is a two degree of freedom system, in which the coupling is nonlinear. Without the coupling the system is linear and comprises of two linear oscillators.

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + (x^2 y - \frac{1}{3}y^3)$$

With scaling of the variables one can think of $H = H_0 + \epsilon H_1$, where $\epsilon H_1 = (x^2 y - \frac{1}{3}y^3)$. Then if $\epsilon = 0$

$$H(x, y, p_x, p_y) = H_0(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) = H_x(x, p_x) + H_y(y, p_y)$$

where $H_x(x, p_x) = \frac{1}{2}p_x^2 + \frac{1}{2}x^2$, $H_y(y, p_y) = \frac{1}{2}p_y^2 + \frac{1}{2}y^2$.

The potential energy

$$V(x, y) = \frac{1}{2}(x^2 + y^2) + (x^2 y - \frac{1}{3}y^3)$$

is shaped like a distorted bowl. In polar coordinates

$$V(r, \theta) = \frac{1}{2}r^2 - \frac{1}{3}r^3 \sin(3\theta)$$

The Hamiltonian is independent of time, so energy is conserved.

$$E = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) \geq 0$$

and the motion is confined to lie inside the contour $E = V(x, y)$. There are many different trajectories one may look at, but the best way to understand the dynamics is to look at the Poincaré maps with surface of section $x = 0$. On this section $E \geq p_y^2/2 + V(0, y)$. If there is an integral besides E the successive intersections of a trajectory with the surface of section will fall on a closed curve.

Numerical results show that for lower energies ($E = \frac{1}{12} \Rightarrow \epsilon$ small) most of the trajectories execute regular motion. For higher energies there are large chaotic regions.