## Section 1.3 Canonical Transformations

A canonical transformation is a phase space coordinate transformation and an associated transformation of the Hamiltonian, in which the dynamics given by Hamilton's equations of motion in the two representations describe the same evolution of the system. Canonical transformations can be used to reformulate a problem in coordinates that are easier to understand or that expose some symmetry of the system.

**Time independent Canonical transformations.**

We assume that the old variables $(\mathbf{q}, \mathbf{p})$ satisfy Hamilton's equations of motion $\dot{\mathbf{q}} = \{q, H\}_{\{q,p\}}$ and $\dot{\mathbf{p}} = \{p, H\}_{\{q,p\}}$. Then a canonical transformation preserves this structure.

Suppose the new canonical variables are

$$\mathbf{Q}(\mathbf{q}, \mathbf{p}), \, \mathbf{P}(\mathbf{q}, \mathbf{p})$$

then we will show that the transformation is canonical if the Poisson bracket of the new phase space variables with respect to the old phase space variables satisfies

$$\{Q_j, P_k\} = \delta_{jk} = \left\{ \begin{array}{l} 1 \text{ if } j = k \\ 0 \text{ otherwise} \end{array} \right]$$

$$\{Q_j, Q_k\} = 0 \quad \text{and} \quad \{P_j, P_k\} = 0$$

In this case the Poisson bracket of any two functions is invariant, that is it is the same whether taken as a function of the old variables $(\mathbf{q}, \mathbf{p})$ or the new variables $(\mathbf{Q}, \mathbf{P})$.

$$\{F, G\}_{\{q,p\}} = \sum_{j=0}^{n} \left( \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right)$$

Now using the fact that

$$\frac{\partial F}{\partial q_j} = \sum_{k=0}^{n} \left( \frac{\partial F}{\partial Q_k}\frac{\partial Q_k}{\partial q_j} + \frac{\partial F}{\partial P_k}\frac{\partial P_k}{\partial q_j} \right)$$

and similar relationships for $\frac{\partial G}{\partial p_j}$, $\frac{\partial F}{\partial p_j}$ and $\frac{\partial G}{\partial q_j}$ and reorganising the terms gives

$$\{F, G\}_{\{q,p\}} = \sum_{k,\,\ell=0}^{n} \left( \frac{\partial F}{\partial Q_k}\frac{\partial G}{\partial Q_\ell}\{Q_k, Q_\ell\} + \frac{\partial F}{\partial Q_k}\frac{\partial G}{\partial P_\ell}\{Q_k, P_\ell\} + \frac{\partial F}{\partial P_k}\frac{\partial G}{\partial Q_\ell}\{P_k, Q_\ell\} + \frac{\partial F}{\partial P_k}\frac{\partial G}{\partial P_\ell}\{P_k, P_\ell\} \right)$$

$$\{F, G\}_{\{q,p\}} = \sum_{k=0}^{n} \left( \frac{\partial F}{\partial Q_k}\frac{\partial G}{\partial P_k} - \frac{\partial F}{\partial P_k}\frac{\partial G}{\partial Q_k} \right) = \{F, G\}_{\{Q,P\}}$$

Then Hamilton's equations are preserved:

$$\dot{Q}_k = \sum_{j=0}^{n} \left( \frac{\partial Q_k}{\partial q_j} \dot{q}_j + \frac{\partial Q_k}{\partial p_j} \dot{p}_j \right)$$

But from Hamilton's equations of motion

$$\dot{Q}_k = \sum_{j=0}^{n} \left( \frac{\partial Q_k}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial Q_k}{\partial p_j} \frac{\partial H}{\partial q_j} \right) = \{Q_k,\, H\}_{\{q,p\}}$$

Now if we define $\bar{H}(\mathbf{Q},\, \mathbf{P}) = H(\mathbf{q}(\mathbf{Q},\, \mathbf{P}),\, \mathbf{p}(\mathbf{Q},\, \mathbf{P}))$

$$\dot{Q}_k = \{Q_k,\, H\}_{\{q,p\}} = \{Q_k,\, \bar{H}\}_{\{Q,P\}} = \frac{\partial \bar{H}}{\partial P_k}$$

Similarily

$$\dot{P}_k = \{P_k,\, H\}_{\{q,p\}} = \{P_k,\, \bar{H}\}_{\{Q,P\}} = -\frac{\partial \bar{H}}{\partial Q_k}$$

What is remarkable is that the Hamiltonian is not involved, that is that the new Hamiltonian is simply $\bar{H}(\mathbf{Q},\, \mathbf{P}) = H(\mathbf{q}(\mathbf{Q},\, \mathbf{P}),\, \mathbf{p}(\mathbf{Q},\, \mathbf{P}))$. (This is not the case for time dependent canonical transformations. )

Suppose we wanted to transform the linear pendulum

$$H = \frac{p^2}{2} + \omega_0^2 \frac{q^2}{2}$$

to polar like coordinates, $(\theta,\, I)$, that are canonical. Then we can specify the new position variables $\theta$ dependence, say the old momentum is proportional to $\cos\theta$ and the old position to $\sin\theta$ with the amplitude some function of $H$, which we shall assume is a function of the new momenta $I$.

$$q = \frac{\sqrt{2H(I)}}{\omega_0} \sin\theta, \qquad p = \sqrt{2H(I)} \cos\theta$$

For the transformation to be canonical the new coordinates, $(\theta,\, I)$ must satisfy

$$\{\theta,\, I\}_{\{q,p\}} = 1$$

Simply doing the calculation gives

$$\{\theta,\, I\}_{\{q,p\}} = \frac{\omega_0}{H'(I)} \qquad \Rightarrow \qquad \frac{\omega_0}{H'(I)} = 1 \qquad \Rightarrow \qquad H = \omega_0 I$$

or $I = H/\omega_0$ for the momentum variable to be a canonical coordinate.

$$q = \frac{\sqrt{2\omega_0 I}}{\omega_0} \sin\theta, \qquad p = \sqrt{2\omega_0 I} \cos\theta$$

In the transformed variables the Hamiltonian is only a function of the momentum, $H(I)$, so that since $H$ is a constant $I$ is a constant of the motion. this transformation is rather special and the new coordinates are known as action angle variables.

**Properties of Canonical Transformations.**

- **1. Canonical Transformations are symplectic.** In other words the matrix of their derivatives is a symplectic matrix.
- **2. Canonical Transformations preserve Phase Space volume.**
- **3. Generating Functions.** Generating Functions make it much easier to perform Canonical Transformations.

**1. Canonical Transformations are symplectic.**

Consider the matrix of derivatives using the traditional ordering of the variables: $\mathbf{x} = (q_1,\, q_2,\, ...p_1,\, p_2,\, ..)$ then

$$\text{if} \quad \mathbf{X} = (Q_1, Q_2, \cdots, P_1,\, P_2, \cdots) \quad \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = A = \begin{pmatrix} \partial_1 Q & \partial_2 Q \\ \partial_1 P & \partial_2 P \end{pmatrix}$$

which is in block form, where $\partial_1 Q$ is an $n \times n$ block with entries $\frac{\partial Q_i}{\partial q_j}$, $\partial_2 Q$ has entries $\frac{\partial Q_i}{\partial p_j}$, $\partial_1 P$ has entries $\frac{\partial P_i}{\partial q_j}$ and $\partial_2 P$ has entries $\frac{\partial P_i}{\partial p_j}$.

Then $A$ is a symplectic matrix, that is it satisfies

$$J_n = A J_n A^T, \qquad \text{where} \quad J_n = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}$$

To see this consider

$$A J_n A^T = \begin{pmatrix} \{Q_i,\, Q_j\} & \{Q_i,\, P_j\} \\ \{P_i,\, Q_j\} & \{P_i,\, P_j\} \end{pmatrix}$$

Then the result follows from the fact that the transformation is canonical:

$$\{Q_j,\, P_k\} = \delta_{jk}, \quad \{Q_j,\, Q_k\} = 0 \quad \text{and} \quad \{P_j,\, P_k\} = 0$$

Take the following example

$$H(q_1,\, q_2,\, p_1,\, p_2) = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \cos(2q_1 + q_2)$$

The fact that $q_1$ and $q_2$ only appear as $(2q_1 + q_2)$ suggests that the system has a symmetry which we could find by moving to a new set of coordinates, one of which is $(2q_1 + q_2)$. So we set

$$Q_1 = (2q_1 + q_2) \quad \text{and} \quad Q_2 = q_2$$

For the transformation to be canonical we require

$$\{Q_1,\, P_1\} = 1 \quad \Rightarrow \quad 2\frac{\partial P_1}{\partial p_1} + \frac{\partial P_1}{\partial p_2} = 1$$

$$\{Q_1,\, P_2\} = 0 \quad \Rightarrow \quad 2\frac{\partial P_2}{\partial p_1} + \frac{\partial P_2}{\partial p_2} = 0$$

$$\{Q_2,\, P_1\} = 0 \quad \Rightarrow \quad \frac{\partial P_1}{\partial p_2} = 0 \quad \text{and} \quad \{Q_2,\, P_2\} = 1 \quad \Rightarrow \quad \frac{\partial P_2}{\partial p_2} = 1$$

These are only satisfied if

$$P_1 = \frac{p_1}{2}, \quad \text{and} \quad P_2 = p_2 - \frac{p_1}{2}$$

So that with $P_1 = \frac{p_1}{2}$, and $P_2 = p_2 - \frac{p_1}{2}$

$$H(Q_1,\, Q_2,\, P_1,\, P_2) = 2P_1^2 + \frac{(P_2 + P_1)^2}{2} + \cos Q_1$$

Since the Hamiltonian is not a function of $Q_2$, by Hamilton's equations of motion $P_2 = p_2 - \frac{p_1}{2}$ is a constant, or an integral of the motion.

An alternative way to calculate the transformation is to use the matrix of derivatives. If we assume that $Q_1 = (2q_1 + q_2)$ and $Q_2 = q_2$ and that the $P$'s are independent of the $q$'s then

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \quad \text{for some } a,\, b,\, c \text{ and } d$$

But $A$ is symplectic so that

$$AJ_nA^T = \begin{pmatrix} 0 & 0 & 2a+b & 2c+d \\ 0 & 0 & b & d \\ -2a-b & -b & 0 & 0 \\ -2c-d & -d & 0 & 0 \end{pmatrix} = J_n \quad \Rightarrow a = \frac{1}{2},\, b = 0,\, c = -\frac{1}{2},\, d = 1$$

Which gives

$$P_1 = \frac{p_1}{2}, \quad \text{and} \quad P_2 = p_2 - \frac{p_1}{2} \quad \text{as before.}$$

With $P_2$ as a constant of the motion

$$H(Q_1,\, Q_2,\, P_1,\, P_2) = \frac{5}{2}(P_1 + \frac{1}{5}P_2)^2 + \frac{2}{5}P_2^2 + \cos Q_1$$

and the equations of motion in these variables are

$$\dot{Q}_1 = \frac{\partial H}{\partial P_1} = 5(P_1 + \frac{1}{5}P_2) \quad \dot{P}_1 = -\frac{\partial H}{\partial Q_1} = \sin Q_1$$

which has critical points at $P_1 = \frac{1}{5}P_2$ and $Q_1 = n\pi$. Since the linearized matrix is

$$\begin{pmatrix} 0 & 5 \\ \cos Q_1 & 0 \end{pmatrix} \quad \Rightarrow \quad det = -5(-1)^n \quad \begin{matrix} \text{saddle for } n \text{ even} \\ \text{center for } n \text{ odd} \end{matrix}$$

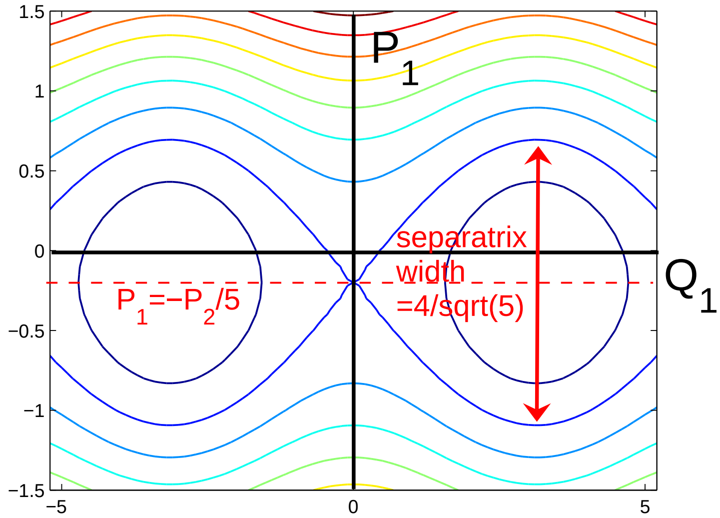
The equation of the separatrix is then $H(Q_1,\, Q_2,\, P_1,\, P_2) = H(0,\, Q_2,\, -\frac{1}{5}P_2,\, P_2) = \frac{2}{5}P_2^2 + 1$, so that

$$1 = \frac{5}{2}(P_1 + \frac{1}{5}P_2)^2 + \cos Q_1$$

From this we can find the width of the separatrix, because at $Q_1 = \pi$ solving for $(P_1 + \frac{1}{5}P_2)$ gives

$$(P_1 + \frac{1}{5}P_2) = \pm\frac{2}{\sqrt{5}} \quad \Rightarrow \quad \Delta P_1 = \frac{4}{\sqrt{5}}$$

The flow in $(Q_1, P_1)$ space is similar to the nonlinear pendulum.




The Poincare Map of this system with surface of section $q_2 = 0$ would give a similar, but "dotted version" of this flow. (The only difference would be the scale on the $q_1$ axis, as we let $Q_1 = 2q_1 + q_2$.)

Because the system has two integrals or constants of the motion, $H$ and $P_2$, the system si said to be **integrable**. The actual solutions for the $Q$'s and the $P$'s will involve elliptic integrals.

**2. Canonical Transformations Preserve Phase Space Volume.**

Suppose the region $R$ in $(\mathbf{q}, \mathbf{p})$ space is transformed to $S$ in $(\mathbf{Q}, \mathbf{P})$ space, then.

$$\int_R \Pi_j dq_j dp_j = \int_S \Pi_j dQ_j dP_j = \int_S \det\ J\ \Pi_j dq_j dp_j$$

where $J$ is the Jacobian of the Transformation. But for a canonical transformation $J = A$ is symplectic and so $detA = 1$ which is true for any symplectic matrix.

$$\int_R \Pi_j dq_j dp_j = \int_S \Pi_j dq_j dp_j$$

### 3. Generating Functions

Both Lagrange's equations of motion and Hamilton's equations of motion can be derived from Hamilton's principle of least action. That is that the realizable path is one that extremises the action

$$S[\mathbf{q}] = \int_{t_0}^{t_1} L(\mathbf{q},\, \dot{\mathbf{q}},\, t)dt$$

or in terms of the Hamiltonian

$$S[\mathbf{q}] = \int_{t_0}^{t_1} (\mathbf{p}\cdot \dot{\mathbf{q}} - H(\mathbf{q},\, \mathbf{p},\, t))\, dt.$$

Suppose now that we transform to new canonical coordinates $(\mathbf{Q},\, \mathbf{P})$. Then the realizable path will also extremise

$$S'[\mathbf{Q}] = \int_{t_0}^{t_1} \Big(\mathbf{P}\cdot \dot{\mathbf{Q}} - K(\mathbf{Q},\, \mathbf{P},\, t)\Big)\, dt,$$

where $K(\mathbf{Q},\, \mathbf{P},\, t)$ is the new Hamiltonian. Note that if the transformation is independent of time then this is just the old Hamiltonian in terms of the new variables, but here we will include the possibility that the transformation is dependent on time.

Now recall that there remained some ambiguity in the Lagrangian we could use. We defined the Lagrangian as the difference between the kinetic and potential energies. But the alternative Lagrangian $L' = L + \frac{dF}{dt}$ for some function $F$ also satisfied Hamilton's principle of least action. Similarly to extremise $S[\mathbf{q}]$ and $S'[\mathbf{Q}]$ does not necessarily imply that the integrands are equal, they may differ by a complete differential.

$$(\mathbf{p}\cdot \dot{\mathbf{q}} - H(\mathbf{q},\, \mathbf{p},\, t)) = \Big(\mathbf{P}\cdot \dot{\mathbf{Q}} - K(\mathbf{Q},\, \mathbf{P},\, t)\Big) + \frac{dF}{dt}$$

There are various sets of variables one can choose $F$ to be a function of. Suppose we choose $F = F_1$ to be a function of the new and old angles.

$$F_1(\mathbf{q},\, \mathbf{Q},\, t): \qquad \text{then} \qquad \frac{dF_1}{dt} = grad_{\mathbf{q}} F_1\cdot \dot{\mathbf{q}} + grad_{\mathbf{Q}} F_1\cdot \dot{\mathbf{Q}} + \frac{\partial F_1}{\partial t}$$

This implies that

$$(\mathbf{p} - grad_{\mathbf{q}} F_1)\cdot \dot{\mathbf{q}} + (K(\mathbf{Q},\, \mathbf{P},\, t) - H(\mathbf{q},\, \mathbf{p},\, t) - \frac{\partial F_1}{\partial t}) = (\mathbf{P} + grad_{\mathbf{Q}} F_1)\, \dot{\mathbf{Q}}$$

which is satisfied if

$$\mathbf{p} = grad_{\mathbf{q}} F_1, \qquad \mathbf{P} = -grad_{\mathbf{Q}} F_1 \qquad \text{and} \qquad K(\mathbf{Q},\, \mathbf{P},\, t) = H(\mathbf{q},\, \mathbf{p},\, t) + \frac{\partial F_1}{\partial t}$$

Alternatively one could choose $F_2(\mathbf{q},\,\mathbf{P},\,t)$, $F_3(\mathbf{Q},\,\mathbf{p},\,t)$ or $F_4(\mathbf{p},\,\mathbf{P},\,t)$. Each type of generating function is useful in a different situation and can be related to $F_1(\mathbf{q},\,\mathbf{Q},\,t)$. The most commonly used is $F_2(\mathbf{q},\,\mathbf{P},\,t) = F_1(\mathbf{q},\,\mathbf{Q},\,t) + \mathbf{P}\cdot\mathbf{Q}(\mathbf{q},\,\mathbf{P})$, which is a Legendre transform away from $-F_1(\mathbf{q},\,\mathbf{Q},\,t)$. (The new variables are active, with $\mathbf{q}$ and $t$ being passive. This means that, with respect to the active variables $\mathbf{P} = -grad_{\mathbf{Q}}F_1$ becomes $\mathbf{Q} = grad_{\mathbf{P}}F_2$ and with respect to the passive variables $\mathbf{p} = grad_{\mathbf{q}}F_1$ becomes $\mathbf{p} = grad_{\mathbf{q}}F_2$.) Alternatively one can substitute in as before

$$(\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\,\mathbf{p},\,t)) = \Big(\mathbf{P}\cdot\dot{\mathbf{Q}} - K(\mathbf{Q},\,\mathbf{P},\,t)\Big) + \frac{d(F_2 - \mathbf{P}\cdot\mathbf{Q})}{dt} = \Big(-\dot{\mathbf{P}}\cdot\mathbf{Q} - K(\mathbf{Q},\,\mathbf{P},\,t)\Big) + \frac{dF_2}{dt}$$

with

$$\frac{dF_2}{dt} = grad_{\mathbf{q}}F_2\cdot\dot{\mathbf{q}} + grad_{\mathbf{P}}F_2\cdot\dot{\mathbf{P}} + \frac{\partial F_2}{\partial t}.$$

This implies that

$$(\mathbf{p} - grad_{\mathbf{q}}F_2)\cdot\dot{\mathbf{q}} + K(\mathbf{Q},\,\mathbf{P},\,t) - H(\mathbf{q},\,\mathbf{p},\,t) - \frac{\partial F_2}{\partial t} = -\dot{\mathbf{P}}\cdot(\mathbf{Q} - grad_{\mathbf{P}}F_2)$$

which is satisfied if

$$\mathbf{p} = grad_{\mathbf{q}}F_2, \qquad \mathbf{Q} = grad_{\mathbf{P}}F_2 \qquad \text{and} \qquad K(\mathbf{Q},\,\mathbf{P},\,t) = H(\mathbf{q},\,\mathbf{p},\,t) + \frac{\partial F_2}{\partial t}.$$

*Example* The identity transformation can be generated by $F_2(\mathbf{q},\,\mathbf{P}) = \mathbf{P}\cdot\mathbf{q}$.

$$\text{If} \quad F_2(\mathbf{q},\,\mathbf{P}) = \sum_{j=1}^{n} P_j q_j \quad \Rightarrow \quad p_j = \frac{\partial F_2}{\partial q_j} = P_j, \quad Q_j = \frac{\partial F_2}{\partial P_j} = q_j.$$

Near identity transformations are also generated by $F_2$ generating functions:

$$F_2(\mathbf{q},\,\mathbf{P},\,t) = \mathbf{P}\cdot\mathbf{q} + \epsilon G(\mathbf{q},\,\mathbf{P},\,t) \quad \text{where } \epsilon \text{ is assumed to be small.}$$

*Example* The example we looked at before

$$H(q_1,\,q_2,\,p_1,\,p_2) = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \cos(2q_1 + q_2)$$

can also be generated by an $F_2$ generating function.

$$\text{We chose} \qquad Q_1 = (2q_1 + q_2) \quad \text{and} \quad Q_2 = q_2$$

So that

$$\frac{\partial F_2}{\partial P_1} = (2q_1 + q_2) \quad \text{and} \quad \frac{\partial F_2}{\partial P_2} = q_2 \quad \Rightarrow \quad F_2 = (2q_1 + q_2)P_1 + q_2 P_2$$

So that

$$p_1 = \frac{\partial F_2}{\partial q_1} = 2P_1 \quad \text{and} \quad p_2 = \frac{\partial F_2}{\partial q_2} = P_1 + P_2 \quad \Rightarrow \quad P_1 = \frac{p_1}{2},\, P_2 = p_2 - \frac{p_1}{2}$$

as before.

*Example* The two body problem.

Consider the motion of two masses $m_1$ and $m_2$, subject only to a mutual gravitational attraction described by the potential $V(r)$, where $r$ is the distance between the particles. Let $\mathbf{x_1}$ and $\mathbf{x_2}$ be the coordinates of the two masses and $\mathbf{p_1}$ and $\mathbf{p_2}$ their conjugate momenta. Then if $r = ||\mathbf{x_1} - \mathbf{x_2}||$ and

$$H(\mathbf{x_1},\, \mathbf{x_2},\, \mathbf{p_1},\, \mathbf{p_2}) = \frac{\mathbf{p_1}\cdot \mathbf{p_1}}{2m_1} + \frac{\mathbf{p_2}\cdot \mathbf{p_2}}{2m_2} + V(r)$$

Now consider new variables

$$\mathbf{x} = \mathbf{x_2} - \mathbf{x_1} \quad \text{and} \quad \mathbf{X} = a\mathbf{x_1} + b\mathbf{x_2} \quad \text{for some } a \text{ and } b$$

Now choose an $F_2$ generating function:

$$F_2(\mathbf{x_1},\, \mathbf{x_2},\, \mathbf{P_1},\, \mathbf{P_2}) : \quad \mathbf{x} = grad_{\mathbf{P_1}} F_2 \quad \text{and} \quad \mathbf{X} = grad_{\mathbf{P_2}} F_2$$

$$\Rightarrow \quad F_2 = \mathbf{P_1}\cdot (\mathbf{x_2} - \mathbf{x_1}) + \mathbf{P_2}\cdot (a\mathbf{x_1} + b\mathbf{x_2})$$

so that $\quad \mathbf{p_1} = -\mathbf{P_1} + a\mathbf{P_2} \quad$ and $\quad \mathbf{p_2} = \mathbf{P_1} + b\mathbf{P_2} \quad \Rightarrow \quad \mathbf{P_1} = \dfrac{a\mathbf{p_2} - b\mathbf{p_1}}{a+b},\ \mathbf{P_2} = \dfrac{\mathbf{p_1} + \mathbf{p_2}}{a+b}$

This means that

$$H(\mathbf{x},\, \mathbf{X},\, \mathbf{P_1},\, \mathbf{P_2}) = \frac{|-\mathbf{P_1} + a\mathbf{P_2}|^2}{2m_1} + \frac{|\mathbf{P_1} + b\mathbf{P_2}|^2}{2m_2} + V(||\mathbf{x}||)$$

$$= \frac{|\mathbf{P_1}|^2}{2\mu} + \frac{|\mathbf{P_2}|^2}{2M} + \left(\frac{b}{m_2} - \frac{a}{m_1}\right) \mathbf{P_1}\cdot \mathbf{P_2}$$

where $\mu = \frac{1}{m_1} + \frac{1}{m_2}$ is called the reduced mass, and $\frac{1}{M} = \frac{a^2}{m_1} + \frac{b^2}{m_2}$. If we choose $a$ and $b$ to remove the $\mathbf{P_1}\cdot \mathbf{P_2}$ term then since $\mathbf{P_2}$ is a constant of the motion which is uncoupled from $\mathbf{P_1}$ and since the only position variable is $r = ||x||$ the system is, as given earlier, effectively a one degree of freedom system

$$H(r,\, p_r,\, p_\theta) = \frac{p_r^2 + p_\theta^2/r^2}{2\mu} + V(r) + \quad \text{constants of the motion.}$$

**Rotating Coordinates** Suppose we have a one degree of freedom time dependent Hamiltonian, where the time and position variable only appear as $q - \omega t$.

$$H(q,\, p,\, t) = \frac{p^2}{2} + V(q - \omega t)$$

Then the time dependence can be 'removed' by moving to new canonical variables via an $F_2$ generating function.

$$\text{Let} \quad Q = q - \omega t, \quad \Rightarrow \quad p = \frac{\partial F_2}{\partial q},\ Q = \frac{\partial F_2}{\partial P} \quad \Rightarrow \quad F_2 = P(q - \omega t).$$

In fact $p$ is unchanged as $\frac{\partial F_2}{\partial q} = P$. But the Hamiltonian is changed. The new Hamiltonian $K(Q,\, P)$ is

$$K(Q,\, P) = H(Q + \omega t,\, P,\, t) + \frac{\partial F_2}{\partial t} = \frac{P^2}{2} + V(Q) - \omega P = \frac{(P-\omega)^2}{2} + V(Q) - \frac{\omega^2}{2}$$

The resulting system is integrable as $K(Q,\, P)$ is independent of time.