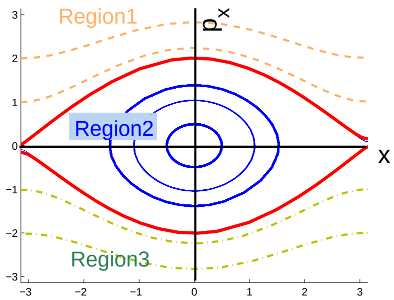
Section 1.4 Integrable Systems and Action Angle variables

If an n degree of freedom system has n independent conserved quantities then the solution to the problem can be reduced to quadratures. Such a system is said to be **Integrable**. Typically the phase space of integrable systems is divided up into regions of similar behaviour separated by solutions asymptotic to the unstable critical points. Take the nonlinear pendulum. The separatrix divides the phase space into three regions.



For any system that is reducible to quadratures a set of phase space coordinates can be chosen for each subregion of the phase space in which the momenta is a constant of the motion. For a one degree of freedom system the momenta is a function of the Hamiltonian. If further the phase space is bounded, as in region 2 for the pendulum, generalized coordinates can be taken as angles and the conjugate momenta are called actions.

Action angle variables can be generated by F_2 generating functions.

The idea of **Action angle variables** in bounded regions is to find new canonical coordinates (θ, I) for which

- a) Each phase curve is labeled uniquely by I, which is constant along a phase curve.
- b) Each point on a phase curve is labeled by a single valued function θ.

So for a One degree of freedom system we require $H(I)$, that is the Hamiltonian is a function of I only. Take the example of the nonlinear pendulum.

$$H = \frac{p^2}{2} + V(q) \quad \text{where} \quad V(q) = \omega_0^2 \cos q$$

We require new canonical variables (θ, I) such that

$$\frac{\partial H}{\partial \theta} = 0, \quad \Rightarrow \quad H(I) \quad \Rightarrow \dot{\theta} = \frac{\partial H}{\partial I} = \omega(I) = \text{ constant}$$

This means that $\theta = \omega(I)t + \theta(0)$.

If we choose θ such that it increases by 2π in each period of the motion the time period will be $T = \dfrac{2\pi}{\omega(I)}$, where $\omega(I)$ is the angular frequency.

The new phase space (θ, I) is cylindrical. $S \times \mathbb{R}$.

$$q(\theta + 2\pi, I) = q(\theta, I) \quad \text{and} \quad p(\theta + 2\pi, I) = p(\theta, I)$$

Although the generating function is an F_2 generating function, so that it is a function of the old positions and the new momenta, it is usually denoted as $S(q, I)$. As for an F_2 generating function

$$p = \frac{\partial S}{\partial q} \quad \text{and} \quad \theta = \frac{\partial S}{\partial I}$$

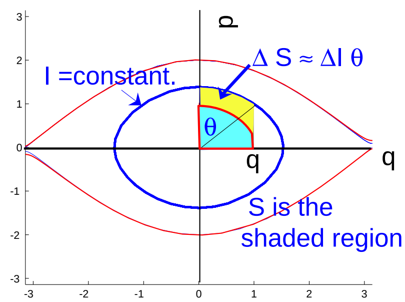
Substituting into the Hamiltonian gives

$$H(q, \frac{\partial S}{\partial q}) = \text{ constant}$$

which is called the Hamiltonian Jacobi equation and it defines S. Here

$$\frac{\partial S}{\partial q} = \pm\sqrt{2\left(H(I) - V(q)\right)} \quad \Rightarrow \quad S = \pm \int_0^q \sqrt{2\left(H(I) - V(q)\right)} dq = \pm \int_0^q p dq$$

Since we assume that I is a constant of the motion it is constant in this integral. Effectively we integrate along a phase curve. this means that there is a geometrical interpretation to S.

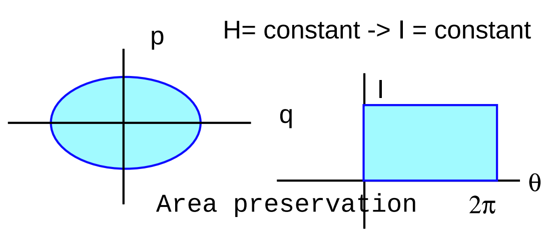


Here rather than specify θ we will find I first, using area preservation.

$$\int_R dq dp = \int_{R'} d\theta dI = \text{ constant}$$

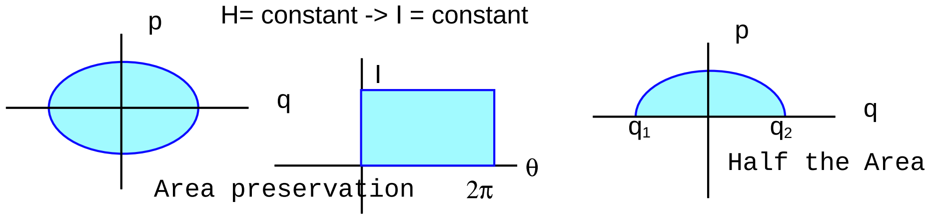
because canonical transformations preserve phase space area, or because the Jacobian of the transformation is 1. Take the area in the transformed variables, where, by design, $I =$ a constant.

$$\int_{R'} d\theta dI = I \int d\theta = 2\pi I \quad \text{for the area inside a phase curve.}$$



From this we can work out $I(H)$.

To work out $I(H)$ use the fact that canonical transformations preserve phase space area and also the reflection symmetry of the Hamiltonian.



$$2\pi I = \int_R dqdp = 2\int_{q_1}^{q_2} pdq = 2\int_{q_1}^{q_2} \sqrt{2(H - V(q))}dq$$

where q_1 and q_2 are the intercepts with the q axis and we have used the reflection symmetry.

So that

$$I = \frac{1}{\pi}\int_{q_1}^{q_2} \sqrt{2(H - V(q))}dq \quad \text{which defines } I(H).$$

In the case where $V(q) = -\omega_0^2 \cos q$, which is an even function

$$I = \frac{2}{\pi}\int_0^{q_0} \sqrt{2(H + \omega_0^2 \cos q)}dq \quad \text{where} \quad H + \omega_0^2 \cos q_0 = 0$$

The solutions are elliptic functions, both inside and outside the separatrix.

On the separatrix $q_0 = \pi$ and $H = \omega_0^2$.

Inside the separatrix $q_0 < \pi$ and $H < \omega_0^2$.

Outside the separatrix $q_0 = \pi$ and $H > \omega_0^2$.

Elliptic Integrals.

The complete elliptic integral of the first kind is

$$F\left(\frac{\pi}{2}; k\right) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} = \int_0^1 \frac{dv}{\sqrt{(1 - v^2)(1 - k^2 v^2)}}$$

where $v = \sin\phi$ so that $dv = \cos\phi d\phi$.

The complete elliptic integral of the second kind is

$$E\left(\frac{\pi}{2}; k\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\phi}d\phi = \int_0^1 \frac{\sqrt{(1 - k^2 v^2)}}{\sqrt{(1 - v^2)}}dv$$

Using half angle formulas,$\cos q = 1 - 2\sin^2\frac{q}{2}$, we can write $I(H)$ outside as

$$I(H) = \frac{2}{\pi}\int_0^{q_0} \sqrt{2(H + \omega_0^2 - 2\omega_0^2 \sin^2\frac{q}{2})}dq = \frac{4}{\pi}\sqrt{2(H + \omega_0^2)}\int_0^{\frac{q_0}{2}} \sqrt{1 - \frac{\sin^2\phi}{k^2}}d\phi$$

where $k^2 = \dfrac{(H + \omega_0^2)}{2\omega_0^2}$

So that

$$I(H) = \frac{8\omega_0 k}{\pi} E\left(\frac{\pi}{2}; \frac{1}{k}\right) \qquad \textbf{Outside the separatrix}$$

Inside the separatrix $k < 1$ and we make the substitution $u = \sin\phi$ which gives $du = \cos\phi d\phi$ or $d\phi = \dfrac{du}{\sqrt{1-u^2}}$.

This means that

$$\int_0^{\frac{q_0}{2}} \sqrt{1 - \frac{\sin^2\phi}{k^2}} d\phi = \int_0^{\sin(\frac{q_0}{2})} \frac{\sqrt{1-\frac{u^2}{k^2}}}{\sqrt{1-u^2}} du$$

Now let $v = \frac{u}{k}$ and recall that $H + \omega_0^2 \cos q_0 = 0$ which means that $\sin^2(\frac{q_0}{2}) = k^2$. Then after some rearrangement

$$I(H) = \frac{8\omega_0}{\pi}\left(E\left(\frac{\pi}{2};k\right) - (1-k^2)F\left(\frac{\pi}{2};k\right)\right) \quad \textbf{Inside the separatrix}$$

where $0 < k < 1$.

Now the **nonlinear frequency** is $\omega(H) = \dfrac{dH}{dI}$.

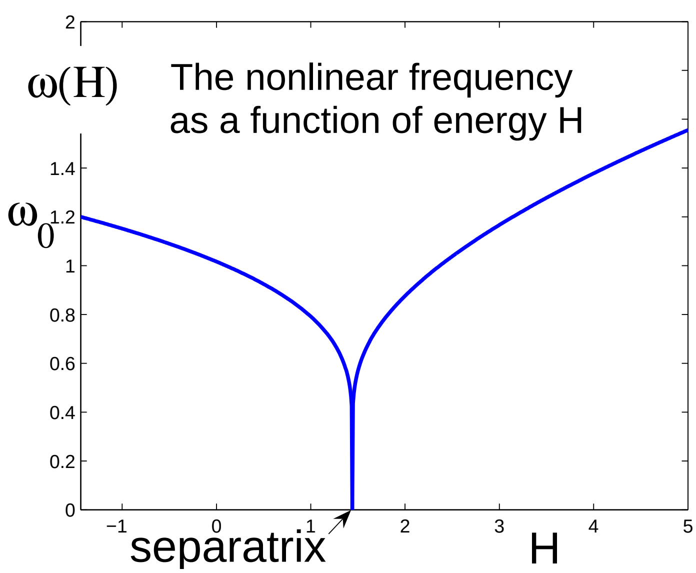
$$\omega(H) = \frac{1}{\frac{dI}{dH}} = \frac{\pi\omega_0}{2}\left\{\begin{array}{ll} \frac{1}{F\left(\frac{\pi}{2};k\right)} & k \leq 1 \text{ inside separatrix.} \\ \frac{k}{F\left(\frac{\pi}{2};\frac{1}{k}\right)} & k > 1 \text{ outside separatrix.} \end{array}\right]$$

Since

$$F\left(\frac{\pi}{2};k\right) \approx \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^4 k^4 + \cdots\right]$$

Note that as $k \to 0$ that is as we move towards the origin $\omega(H) \to \omega_0$.

Also as $k \to \infty$ then $\omega(H) \to k\omega_0$.



The new angle can be found in terms of *in*complete elliptic integrals. Using the fact that $\theta = \dfrac{\partial S}{\partial I}$ and $S = \int_0^q \sqrt{2\left(H + \omega_0^2 \cos q\right)} dq$

$$S = \left\{ \begin{array}{ll} 4\omega_0 k E\left(\phi; \frac{1}{k}\right) & \text{for } k > 1 \\ 4\omega_0 \left(E\left(\phi : k\right) - (1-k^2) F\left(\phi; k\right)\right) & \text{for} k \leq 1 \end{array} \right]$$

where $k \sin\phi = \sin(\frac{q}{2})$.

The inverse elliptic functions can be treated a bit like sin and cos. If

$$u = F\left(\phi : k\right) \quad \text{then} \quad sn(u,\, k) = \sin\phi = \frac{\sin\frac{q}{2}}{k}$$

which defines $q(\theta,\, I)$. But the dependence is quite complicated because $k(I)$.

The momentum $p(\theta,\, I)$, dependence on the new variables is then

$$p = \pm 2\omega_0 \sqrt{k^2 - \sin^2(\frac{q}{2})} = 2\omega_0 \sqrt{k^2 - k^2 sn^2\left(\frac{2F\left(\phi; k\right)\theta}{\pi}, k)\right)}$$

From these formulas it is easy to derive the solutions for $(q,\, p)$ as functions of time. This is because of the simple dependence that $(\theta,\, I)$ have on time.

$$\theta(t) = \omega(I)(t - t_0), \quad \text{where} \quad \omega(I) = \frac{\pi\omega_0}{2F\left(\phi; k\right)}, \quad \text{and} \quad I(t) = I \text{ constant.}$$

So that

$$q = 2 \arcsin\left(k sn(\omega_0(t - t_0), k)\right), \qquad p = 2\omega_0 k cn(\omega_0(t - t_0), k), \quad \text{where} \quad cn(x, k) = \sqrt{1 - sn^2(x, k)}$$

and $k(I)$ is a constant of the motion.

Integrability and Action Angle variables

In terms of action angle variables the conditions for integrability are easy to understand.

- **Complete Integrability**, of an n degree of freedom system, requires n linearly independent constants of the motion, (I_k) for $k = 1 \cdots n$, that satisfy the condition that the Poisson bracket of any two is zero, $\{I_k,\, I_\ell\} = 0$.

If an n degree of freedom system has n linearly independent constants of the motion then we can transform to action angle variables where the new linearly independent constants are the new actions, (I_k) for $k = 1 \cdots n$, if and only if $\{I_k,\, I_\ell\} = 0$.

Note that the (I_k) must be linearly independent because otherwise they will not be able to represent the configuration space, which is n dimensional. For the transformation to be canonical we also require that $\{I_k,\, I_\ell\} = 0$.

Example Take a Hamiltonian that can be separated into linear oscillators.

$$H = \sum_i H_i(q_i,\, p_i,\, t) = \left(\frac{p_1^2}{2} + \omega_1^2 \frac{q_1^2}{2}\right) + \left(\frac{p_2^2}{2} + \omega_2^2 \frac{q_2^2}{2}\right) + \cdots$$

Then $H_i = \left(\frac{p_i^2}{2} + \omega_i^2 \frac{q_i^2}{2}\right)$ are constants of the motion and could be taken as the new momenta, $I_i = H_i$.