Section 1.5 Canonical Perturbation Theory.

Closed form solutions to Hamiltonian systems can be found only rarely. However some systems differ from a solvable system by the addition of a small effect. The goal of perturbation theory is to relate aspects of the motion of a not completely integrable system to those of a nearby integrable system. In effect we will try to find action angle variables for the new system. In so doing we will find that we can use perturbation theory to predict where large resonances regions are located and even to estimate their widths.

I will mainly consider two types of systems:

$1\frac{1}{2}$ **Degree of Freedom systems**, which are easier to grasp but for which we will need time dependent Generating functions, and

2 **Degree of Freedom systems**, whose phase space is $4D$.

Perturbation theory for $1\frac{1}{2}$ Degree of Freedom systems

Suppose that

$$H(q,\, p,\, t) = H_0(q,\, p) + \epsilon H_1(q,\, p,\, t) + \epsilon^2 H_2(q,\, p,\, t) + \cdots$$

where since $H_0(q,\, p)$ is a 1 degree of freedom system action angle variables can be found so that

$$H_0(q,\, p) = H_0(I) \quad \text{for some } I$$

Then in terms of these variables we have

$$H(\theta,\, I,\, t) = H_0(I) + \epsilon H_1(\theta,\, I,\, t) + \epsilon^2 H_2(\theta,\, I,\, t) + \cdots$$

Since the $F_2(\theta,\, J,\, t)$ generating function can generate the identity transformation we will use it to generate a near identity transformation. That is it will generate a canonical transformation that is the identity for $\epsilon = 0$. So assuming F_2 as a series in ϵ

$$F_2(\theta,\, J,\, t) = J\theta + \epsilon G(\theta,\, J,\, t) = J\theta + \epsilon\left(G_1(\theta,\, J,\, t) + \epsilon G_2(\theta,\, J,\, t) + \cdots\right)$$

This gives the following transformation to $(\phi,\, J)$

$$I = \tfrac{\partial F_2}{\partial \theta} = J + \epsilon \tfrac{\partial G_1}{\partial \theta} + \epsilon^2 \tfrac{\partial G_2}{\partial \theta} + \cdots$$

$$\phi = \tfrac{\partial F_2}{\partial J} = \theta + \epsilon \tfrac{\partial G_1}{\partial J} + \epsilon^2 \tfrac{\partial G_2}{\partial J} + \cdots$$

and the new Hamiltonian

$$K(\phi,\, J,\, t) = H_0(I(\phi,\, J,\, t)) + \epsilon H_1(\theta(\phi,\, J,\, t),\, I(\phi,\, J,\, t),\, t) + \epsilon^2 H_2 + \cdots + \frac{\partial F_2}{\partial t}(\theta(\phi,\, J,\, t),\, J,\, t)$$

Now for the moment we will think of this as a function of $(\theta,\, J,\, t)$, where $I = J + \epsilon \frac{\partial G}{\partial \theta}$

$$H_i(\theta,\, I(\theta,\, J,\, t),\, t) = H_i(\theta,\, J,\, t) + \epsilon \frac{\partial H_i}{\partial I}(\theta,\, J,\, t)\frac{\partial G}{\partial \theta} + \frac{\epsilon^2}{2}\frac{\partial^2 H_i}{\partial I^2}(\theta,\, J,\, t)\left(\frac{\partial G}{\partial \theta}\right)^2 + \cdots$$

where

$$\frac{\partial G}{\partial \theta} = \frac{\partial G_1}{\partial \theta} + \epsilon \frac{\partial G_2}{\partial \theta} + \cdots$$

Putting this all together with $I = J + \epsilon \frac{\partial G}{\partial \theta} = J + \epsilon \frac{\partial G_1}{\partial \theta} + \epsilon^2 \frac{\partial G_2}{\partial \theta} + \cdots$ this becomes

$$K = H_0(J) + \epsilon \frac{\partial H_0}{\partial I}(J) \left(\frac{\partial G_1}{\partial \theta} + \epsilon \frac{\partial G_2}{\partial \theta} + \cdots \right) + \frac{\epsilon^2}{2} \frac{\partial^2 H_0}{\partial I^2}(J) \left(\frac{\partial G_1}{\partial \theta} + \cdots \right)^2 + \cdots$$

$$+\epsilon H_1(\theta,\, J,\, t) + \epsilon^2 \left(\frac{\partial H_1}{\partial I}(\theta,\, J,\, t) \frac{\partial G_1}{\partial \theta} + \cdots \right) + \cdots$$

$$+\epsilon^2 H_2(\theta,\, J,\, t) + \cdots$$

$$+\epsilon \frac{\partial G_1}{\partial t} + \epsilon^2 \frac{\partial G_2}{\partial t} + \cdots$$

which becomes

$$K = H_0(J) + \epsilon \left(\frac{\partial H_0}{\partial I}(J) \frac{\partial G_1}{\partial \theta} + H_1(\theta,\, J,\, t) + \frac{\partial G_1}{\partial t} \right)$$

$$+\epsilon^2 \left(\frac{\partial H_0}{\partial I}(J) \frac{\partial G_2}{\partial \theta} + \frac{1}{2} \frac{\partial^2 H_0}{\partial I^2}(J) \left(\frac{\partial G_1}{\partial \theta} \right)^2 + \frac{\partial H_1}{\partial I} \frac{\partial G_1}{\partial \theta} + H_2(\theta,\, J,\, t) + \frac{\partial G_2}{\partial t} \right) + \cdots$$

The idea is then to choose G_i to remove as many oscillating terms as possible. If it is possible to remove all the θ and t dependence by choosing appropriate G_i then the final Hamiltonian is a function only of the new action, $K(J)$. So the final Hamiltonian is integrable.

Take a *simple example*, say

$$H(\theta,\, I) = H_0(I) + \epsilon\, (\cos(\theta) + I \sin t) \qquad \text{so that} \quad H_1(\theta,\, I,\, t) = \cos(\theta) + I \sin t$$

Then choose $G_i(\theta,\, J,\, t)$ to remove the terms of order i in ϵ.
So choose $G_1(\theta,\, J,\, t))$ such that

$$\frac{\partial H_0}{\partial I}(J) \frac{\partial G_1}{\partial \theta} + H_1 + \frac{\partial G_1}{\partial t} = 0$$

$$\Rightarrow \qquad \frac{\partial H_0}{\partial I}(J) \frac{\partial G_1}{\partial \theta} + \frac{\partial G_1}{\partial t} + \cos(\theta) + J \sin t = 0$$

So that

$$G_1(\theta,\, J,\, t) = -\frac{\sin(\theta)}{\omega(J)} + J \cos t \qquad \text{where} \omega(J) = \frac{\partial H_0}{\partial I}(J) \neq 0$$

Then choose $G_2(\theta,\, J,\, t)$ to remove any oscillating terms or order ϵ^2, that is set

$$\frac{\partial H_0}{\partial I} \frac{\partial G_2}{\partial \theta} + \frac{1}{2} \frac{\partial^2 H_0}{\partial I^2} \left(\frac{\partial G_1}{\partial \theta} \right)^2 + \frac{\partial H_1}{\partial I} \frac{\partial G_1}{\partial \theta} + \frac{\partial G_2}{\partial t} = 0$$

Note here $H_2 = 0$.

If this can be done to all orders then the new Hamiltonian is only a function of J, $K(J)$, and the system is integrable.

Here if $H_1(\theta, I) = \cos(\theta)$ only we would be able to remove all the oscillating terms to all orders, provided $\omega(J) \neq 0$ for any J. So say $H_0(I) = I^2/2$ then $\omega(I) = I$, which if it is assumed to be positive will never equal zero.

Note that, even after removing all the oscillating terms, the resulting Hamiltonian may not simply be $H_0(J)$. For instance here the order ϵ^2 term

$$\frac{1}{2}\frac{\partial^2 H_0}{\partial I^2}\left(\frac{\partial G_1}{\partial \theta}\right)^2 = \frac{1}{4}\omega'(J)(1 + \cos(2\theta))$$

has a non oscillating part. $\omega'(J)/4$ is not a function of θ.

To order ϵ^2 the new Hamiltonian is

$$K(J) = H_0(J) + \epsilon^2 \frac{1}{4}\omega'(J).$$

which is a function of J only. To this order $J = I + \epsilon \frac{\cos\theta}{\omega(I)}$ is the approximate action for the system.

Also if $H_1(\theta, I) = I \sin t$ we would be able to remove the oscillating terms at all orders.

However if both terms are included we get problems with small divisors further down the track.

Small Divisors

To see the problem of small divisors consider $H_1(\theta, I, t) = \cos(\theta) + I \sin t$, then as before the oscillating terms of order ϵ can be removed by a near identity transform with $G_1 = -\frac{\sin(\theta)}{\omega(J)} + J \cos t$.

The procedure is recursive, so at order ϵ^2 we must choose G_2 to remove the oscillating terms at this order, which since $H_2 = 0$ here are simply

$$\frac{1}{2}\frac{\partial^2 H_0}{\partial I^2}\left(\frac{\partial G_1}{\partial \theta}\right)^2 + \frac{\partial H_1}{\partial I}\frac{\partial G_1}{\partial \theta}$$

Or

$$\begin{aligned}\frac{1}{2}\frac{\partial \omega}{\partial I}(J)\left(\frac{\partial G_1}{\partial \theta}\right)^2 + \frac{\partial H_1}{\partial I}(J)\frac{\partial G_1}{\partial \theta} &= \frac{\omega'(J)}{2\omega^2(J)}\cos^2\theta - \frac{\sin(t)\sin(\theta)}{\omega(J)} \\ &+ \frac{\omega'(J)}{4\omega^2(J)}(1 + \cos(2\theta)) - \frac{1}{2\omega(J)}(\cos(\theta + t) + \cos(\theta - t))\end{aligned}$$

To remove the oscillating terms set

$$\omega(J)\frac{\partial G_2}{\partial \theta} + \frac{\partial G_2}{\partial t} = -\frac{\omega'(J)}{4\omega^2(J)}\cos(2\theta) + \frac{1}{2\omega(J)}(\cos(\theta + t) + \cos(\theta - t))$$

Solving for G_2 gives

$$G_2 = -\frac{\omega'(J)}{4\omega^3(J)}\sin(2\theta) + \frac{\sin(\theta + t)}{2\omega(J)(1 + \omega(J))} + \frac{\sin(\theta - t)}{2\omega(J)(-1 + \omega(J))}$$

Now we see the problem of small divisors, for this is only possible if $\omega(J) \neq \pm 1$. If it is the case that $\omega(J) \neq \pm 1$ then we can proceed to solve for G_3. However this involves further small divisors because of terms of the form $\sin(\theta \pm 2t)$, which we can only remove if $\omega(J) \neq \pm 2$, and terms of the form $\sin(2\theta \pm t)$, which we can only remove if $2\omega(J) \neq \pm 1$ etc.

The problem is reminiscent of resonances in forced linear systems and is hard to avoid in forced nonlinear systems.

It is instructive, at this point, to consider which Hamiltonians we can so far prove to be integrable.

For instance a time dependent 1 degree of freedom Hamiltonian system, where the time dependence is a total derivative is integrable. This follows from the fact that $H' = H + \frac{dF}{dt}(t)$ is an alternative Hamiltonian for the system. But one could also imagine a canonical transformation where $\frac{\partial F}{\partial t} = \frac{dF}{dt}(t)$. So for instance

$$H(q,\, p,\, t) = H_0(q,\, p) + h(t) \quad \text{is an integrable system}$$

Also if all the position and time dependence is a function of one variable, say $(\omega t + \Omega\theta)$ in a $1\frac{1}{2}$ d.o.f system or say $(\omega_1\theta_1 + \omega_2\theta_2)$ in a 2 d.o.f system, then the system is integrable. This follows from the fact that you can always transform to an integrable system via new variables, $\phi = (\omega t + \Omega\theta)$ or $\phi = (\omega_1\theta_1 + \omega_2\theta_2)$. So

$$H(q,\, p,\, t) = H_0(I) + H_1(\theta - \alpha t,\, I) \quad \text{and} \quad H(\theta_1,\, \theta_2,\, I_1,\, I_2) = H_0(I_1,\, I_2) + H_1(\theta_1 - \alpha\theta_2,\, I_1,\, I_2),$$

where α is a constant, are integrable.

The Parametrically Excited Nonlinear Pendulum, is not integrable.

$$H(\phi,\, p,\, t) = \frac{p^2}{2} - \omega_0^2(1 + \epsilon\cos(\omega t))\cos\phi = H_0(\phi,\, p) + \epsilon H_1(\phi,\, p,\, t)$$

gives

$$H_0(\phi,\, p) = \frac{p^2}{2} + \omega_0^2\cos\phi \quad \text{and} \quad H_1(\phi,\, p,\, t) = -\omega_0^2\cos(\omega t)\cos\phi = -\frac{\omega_0^2}{2}(\cos(\omega t + \phi) + \cos(\omega t - \phi))$$

Now we know that for $\epsilon = 0$ the nonlinear frequency inside the separatrix is given by

$$\omega(I) = \frac{\pi\omega_0}{2F(\frac{\pi}{2};\, k)} \qquad \text{where} \quad k^2 = \frac{\omega_0^2 + H_0}{2\omega_0^2}$$

So, from the form of H_1 we might expect to have problems with a perturbation scheme if $\dfrac{\omega}{\omega(I)}$ is a rational. Consider

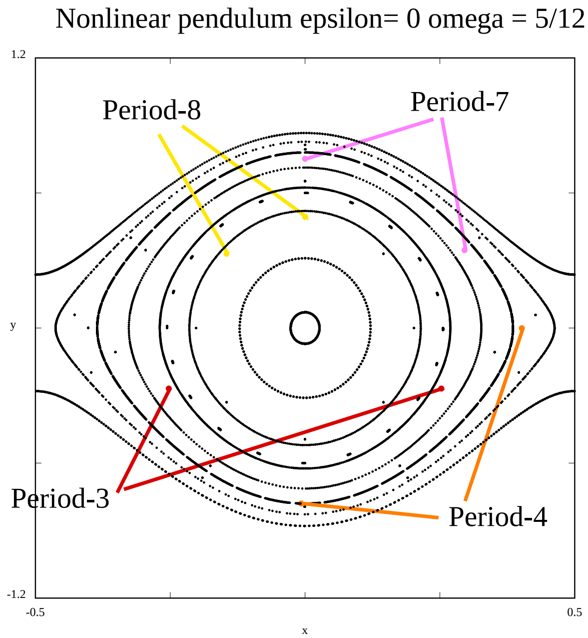
$$\frac{\omega}{\omega(I)} = \frac{1}{3} \quad \Rightarrow \quad F(\frac{\pi}{2};\, k) = \frac{3\pi}{2}\omega_0$$

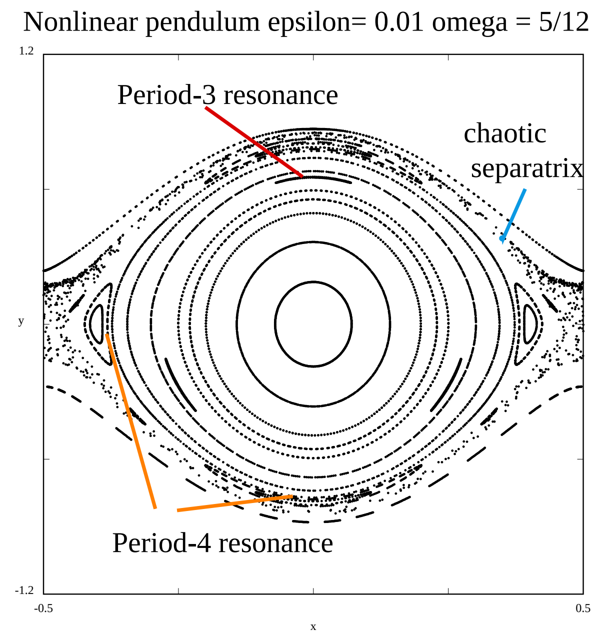
from which we can solve for k^* and hence for $H_0 = H_0^*$.

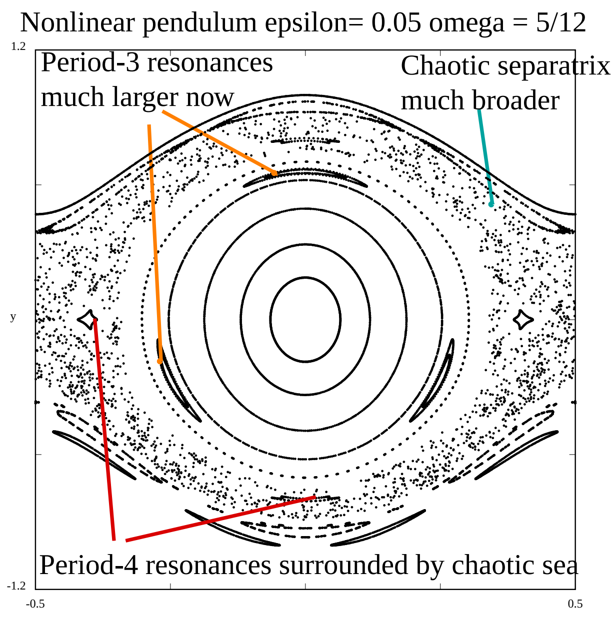
To see this suppose we are on $H_0 = H_0^*$, consider the point on the momentum axis, whose position is given by $p_0 = \sqrt{2(H_0^* + \omega_0^2)} = 2\omega_0 k^*$, then the Poincare Map should give just three points, that is a period-3 orbit. (This is worth trying yourself.) Here are some of the values where ω is taken as 1 and $\omega_0 = \frac{5}{12}$:

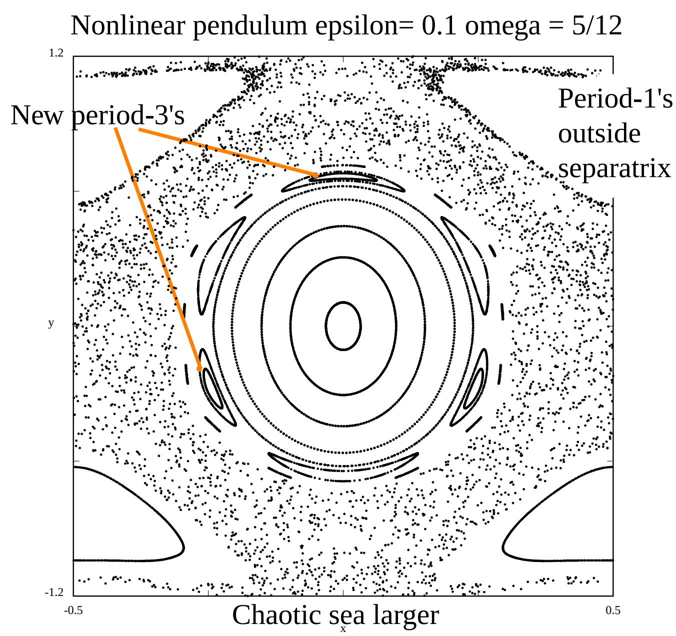
$$\begin{array}{ccc} \omega(I) = \frac{1}{3} & \Rightarrow & p_0 = 0.652276 \\ \omega(I) = \frac{1}{4} & \Rightarrow & p_0 = 0.79412 \\ \omega(I) = \frac{3}{8} & \Rightarrow & p_0 = 0.494089 \\ \omega(I) = \frac{2}{7} & \Rightarrow & p_0 = 0.751941 \\ \omega(I) = \frac{2}{9} & \Rightarrow & p_0 = 0.813806 \end{array}$$

Now if you start at the same points, but set $\epsilon \neq 0$, but small, you will often find that island resonances have formed. The actual resonances depend on the actual perturbation and we need action angle variables to calculate them..


Nonlinear pendulum epsilon= 0 omega = 5/12
Period-8
Period-7
Period-3
Period-4
1.2
-1.2
y
-0.5
0.5
x


Nonlinear pendulum epsilon= 0.01 omega = 5/12
Period-3 resonance
chaotic
separatrix
Period-4 resonance
1.2
-1.2
y
-0.5
0.5
x


Nonlinear pendulum epsilon= 0.05 omega = 5/12
Period-3 resonances
much larger now
Chaotic separatrix
much broader
Period-4 resonances surrounded by chaotic sea
1.2
-1.2
y
-0.5
0.5
x


Nonlinear pendulum epsilon= 0.1 omega = 5/12
New period-3's
Period-1's
outside
separatrix
Chaotic sea larger
1.2
-1.2
y
-0.5
0.5
x

Resonances for the Parametrically Excited Nonlinear Pendulum

Using action angle variables, as described in the last section of the notes, for inside the separatrix

$$I(H) = \frac{8\omega_0}{\pi}\left(E\left(\frac{\pi}{2};k\right) - (1-k^2)F\left(\frac{\pi}{2};k\right)\right), \quad \omega(I) = \frac{\pi\omega_0}{2}\frac{1}{F\left(\frac{\pi}{2};k\right)}$$

where $k = \sqrt{\frac{H_0+\omega_0^2}{2\omega_0}}$ and $q(\theta, I)$ and $p(\theta, I)$ are know functions of (θ, I), we can rewrite the Hamiltonian as

$$H(\theta, I, t) = H_0(I) + \epsilon H_1(\theta, I, t) \qquad \text{where } H_0(I) \text{ is given above}$$

Then

$$H_1(\theta, I, t) = \omega_0^2 \cos(\omega t)\left(1 - 2k^2 sn^2\left(\frac{2F\left(\frac{\pi}{2};k\right)\theta}{\pi}, k\right)\right)$$

Now the elliptic function (sn) can be expanded as a fourier series in θ.

$$k\ \ sn\left(\frac{2F\theta}{\pi}\right) = \frac{2\pi}{F}\sum_{m=0}\frac{\bar{q}^{m+\frac{1}{2}}}{(1-\bar{q}^{2m+1})}\sin((2m+1)\theta) \quad \text{where} \quad \bar{q} = e^{\frac{\pi F\left(\frac{\pi}{2};\sqrt{1-k^2}\right)}{F\left(\frac{\pi}{2};k\right)}}$$

So we can expand H_1 as a series in terms of the form $(\cos(\omega t - 2m\theta) + \cos(\omega t + 2m\theta))$ for $m = 0, 1, 2, ...$

Then

$$H_1(\theta, I, t) = \sum_{m=0,1,\cdots} V_m(I)\left(\cos(\omega t - 2m\theta) + \cos(\omega t + 2m\theta)\right)$$

where

$$V_0 = -\omega_0^2\left(\frac{1}{2} + \frac{4\pi^2}{F^2}\left(\frac{\bar{q}}{(1-\bar{q})^2} + \frac{\bar{q}^3}{(1-\bar{q}^3)^2} + \cdots\right)\right)$$

$$V_1 = -\frac{4\omega_0^2\pi^2}{F^2}\left(-\frac{2\bar{q}^2}{(1-\bar{q})} + \frac{2\bar{q}^3}{(1-\bar{q})(1-\bar{q}^3)} + \cdots\right)$$

$$V_2 = -\frac{4\omega_0^2\pi^2}{F^2}\left(\frac{2\bar{q}^2}{(1-\bar{q})(1-\bar{q}^3)} + \frac{2\bar{q}^3}{(1-\bar{q})(1-\bar{q}^5)} + \cdots\right)$$

Unfortunately all the terms are oscillating so we would like to remove them all.

But this cannot be done for all values of I because of small divisors. In order to remove the oscillating terms at order ϵ we need to choose G_1 such that

$$\frac{\partial H_0}{\partial I}(J)\frac{\partial G_1}{\partial \theta} + H_1 + \frac{\partial G_1}{\partial t} = 0$$

where H_1 is as above. If this can be done, at least for some I, we can go to the order ϵ^2 terms and remove the the θ and t dependence in these terms using G_2, etc.

What we will find is that for some values of I we cannot solve for some G_i due to the problem of small divisors. However there are values of I, specifically where $\frac{\omega(I)}{\omega}$ is sufficiently irrational, where, for ϵ small enough, this can be done and this is the subject of the KAM theorem.

First Order Resonances for the Nonlinear Pendulum.

Using the Fourier series representation of H_1 we let

$$G_1 = \sum_{m=0,\,1,\,\cdots} \left(G_{1m-}(J)\sin(\omega t - 2m\theta) + G_{1m+}(J)\sin(\omega t + 2m\theta)\right) \quad \text{for some } G_{1m-}(J),\, G_{1m+}(J)$$

Then

$$\frac{\partial H_0}{\partial I}(J)\frac{\partial G_1}{\partial \theta} + \frac{\partial G_1}{\partial t}$$

$$= \sum_{m=0,\,1,\,\cdots} \left((\omega - 2m\omega(J))G_{1m-}(J)\cos(\omega t - 2m\theta) + (\omega + 2m\omega(J))G_{1m+}(J)\cos(\omega t + 2m\theta)\right)$$

So if we choose

$$G_{1m-}(J) = -\frac{V_m(J)}{(\omega - 2m\omega(J))} \quad \text{and} \quad G_{1m+}(J) = -\frac{V_m(J)}{(\omega + 2m\omega(J))}$$

then to order ϵ^2

$$K = H(J) + 0(\epsilon^2)$$

and we can proceed to the next level.

However there are problems with the asymptotic series if $(\omega \pm 2m\omega(J)) = 0$ and even if $(\omega \pm 2m\omega(J)) \approx 0$. In fact there are first order resonances if

$$\frac{\omega(I)}{\omega} = \pm\frac{1}{2m} \quad \text{these are the first order resonance conditions.}$$

If $\frac{\omega(I)}{\omega} = \pm\frac{1}{2m}$ we cannot remove the oscillating terms at first order. In fact the integral I given by the resonance condition is resonant with the perturbation.

To see what happens at $(\omega \pm 2m\omega(J)) \approx 0$ imagine we have removed all the resonances except the one at $m = 1$, that is at $(\omega \pm 2\omega(J)) \approx 0$ and that J is approximately given by $(\omega \pm 2\omega(J)) = 0$. Then the Hamiltonian to first order in ϵ is

$$K = H_0(J) + \epsilon V_1(J)\cos(\omega t - 2\theta) + 0(\epsilon^2)$$

Concentrating on the order ϵ terms, it is useful to transform to a rotating frame. So let

$$\psi = \theta - \frac{\omega t}{2} \quad \text{and use an } F_2(J',\, \theta) \text{ generating function}$$

So

$$J = \frac{\partial F_2}{\partial \theta} \quad \text{and} \quad \psi = \frac{\partial F_2}{\partial J'}$$

$$\Rightarrow \quad F_2 = J'(\theta - \frac{\omega t}{2}) \quad \Rightarrow \quad J = J'$$

and the new Hamiltonian is

$$\bar{K} = K + \frac{\partial F_2}{\partial t} = K - \frac{\omega J}{2}$$

$$\bar{K} = H_0(J) - \frac{\omega J}{2} + \epsilon V_1(J)\cos(2\psi) + 0(\epsilon^2)$$

This is only valid away from all the other resonances, so for $J \approx J_0$, where J_0 is given by $(\omega - 2\omega(J_0)) = 0$. So assume that

$$J = J_0 + \sqrt{\epsilon}\Delta J \quad \text{where} \quad \frac{\omega(J_0)}{\omega} = \frac{1}{2}$$

Then since

$$H_0(J) = H_0(J_0) + \sqrt{\epsilon}\frac{\partial H_0}{\partial J}(J_0)\Delta J + \frac{\epsilon}{2}\frac{\partial^2 H_0}{\partial J^2}(J_0)\Delta J^2 + 0(\epsilon^{\frac{3}{2}})$$

$$\bar{K} = H_0(J_0) - \frac{\omega J_0}{2} + \sqrt{\epsilon}\left(\frac{\partial H_0}{\partial J}(J_0) - \frac{\omega}{2}\right)\Delta J + \epsilon\left(\frac{1}{2}\frac{\partial^2 H_0}{\partial J^2}(J_0)\Delta J^2 + V_1(J_0)\cos(2\psi)\right) + 0(\epsilon^{\frac{3}{2}})$$

But $\frac{\partial H_0}{\partial J}(J_0) = \omega(J_0) = \frac{\omega}{2}$ so the order $\sqrt{\epsilon}$ terms vanish. Also the first two terms $H_0(J_0) - \frac{\omega J_0}{2}$ are constant. So setting $\bar{\bar{K}} = \bar{K} - H_0(J_0) + \frac{\omega J_0}{2}$ gives

$$\bar{\bar{K}} = \epsilon\left(\frac{1}{2}\frac{\partial^2 H_0}{\partial J^2}(J_0)\Delta J^2 + V_1(J_0)\cos(2\psi)\right) + 0(\epsilon^{\frac{3}{2}})$$

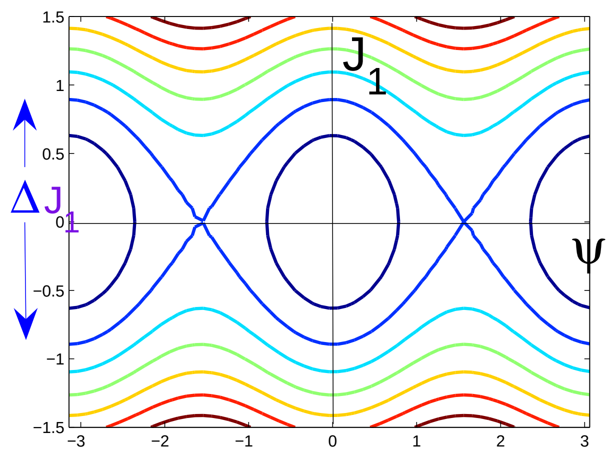
which is essentially the nonlinear pendulum. The only slight unknown is that the cos term is a function of (2ψ) as opposed to ψ. The equations of motion are

$$\dot{\psi} = \epsilon\frac{\partial^2 H_0}{\partial J^2}\Delta J \qquad \dot{\Delta J} = 2\epsilon V_1(J_0)\sin(2\psi)$$

If $V_1(J_0) < 0$ the origin is a center and the separatrix is given by $\bar{\bar{K}} = -\epsilon V_1$.

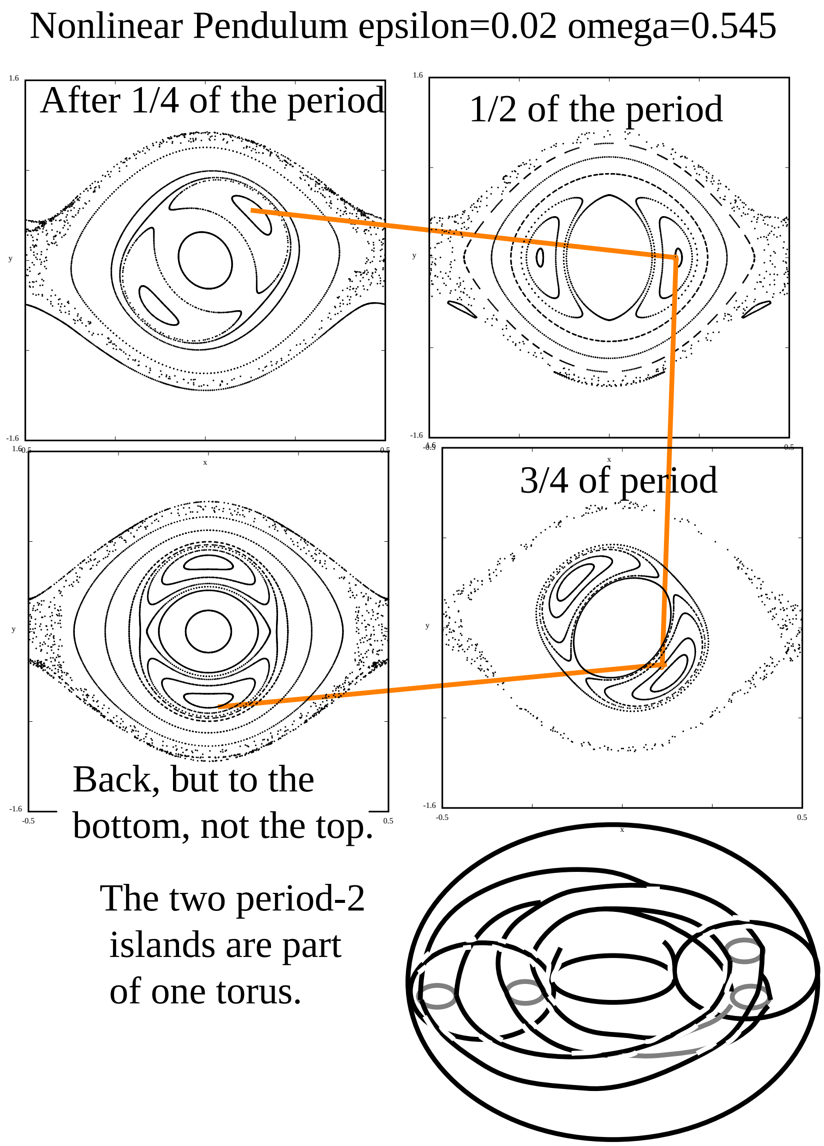
The approximate width of the separatrix is order $\sqrt{\epsilon}$.

$$\Delta J = 4\sqrt{-\frac{V_1(J_0)}{\frac{\partial^2 H_0}{\partial J^2}(J_0)}}$$



In the Poincare Map of the original system this becomes wrapped around the origin. However the order ϵ^2 terms destroy the separatrix so that two regular islands are surrounded by a chaotic sea.

In the full time dependent system the resonances rotate in time. This is because $\psi = \theta - \frac{\omega t}{2}$, so that the critical point $\psi = \psi_0$ moves in time as $\theta = \frac{\omega t}{2} + \psi_0$. After two periods, $t = 2 \times \frac{2\pi}{\omega}$ the angle θ returns.

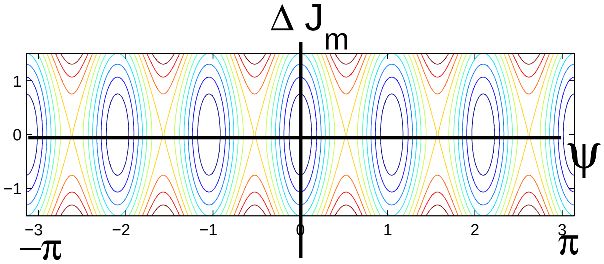


Other first order resonances.

The analysis of other first order resonances is very similar. Once again assume that $J \approx J_{m0}$, where $\frac{\omega(J_{m0})}{\omega} = \frac{1}{2m}$. Let

$$J = J_{m0} + \sqrt{\epsilon}\Delta J_m \quad \text{and} \bar{\bar{K}} = \epsilon\left(\frac{1}{2}\frac{\partial^2 H_0}{\partial J^2}\Delta J_m^2 + V_m(J_{m0})\cos(2m\psi)\right) + 0(\epsilon^{\frac{3}{2}})$$

The result is that there are now $4m$ critical points, $2m$ saddles and $2m$ centers. This period-$2m$ resonance results in a period-$2m$ island chain.



Second Order Resonances

Suppose I is not close to any first order resonances. That is I is not close to I_{m0}, where $\frac{\omega(I_{m0})}{\omega} = \frac{1}{2m}$. Then we can use G_1 to remove all the oscillating terms in H_1. Here this means removing all of H_1. However, even if the orginal Hamiltonian does not constain terms at order ϵ^2, the generating function introduces them! So the new Hamiltonian is

$$K = H_0(J) + \epsilon^2\left(\frac{1}{2}\frac{\partial\omega(J)}{\partial J}\left(\frac{\partial G_1}{\partial\theta}\right)^2 + \frac{\partial H_1}{\partial J}\frac{\partial G_1}{\partial\theta}\right) + 0(\epsilon^3)$$

$$K = H_0(J) + \epsilon^2\frac{1}{2}\frac{\partial\omega(J)}{\partial J}\left(\sum_{m=0,\,1,\,\cdots}(-2mG_{1m-}(J)\cos(\omega t - 2m\theta) + 2mG_{1m+}(J)\cos(\omega t + 2m\theta))\right)^2$$

$$+\epsilon^2\sum_{m=1,\,2..}\frac{dV_m}{dJ}(\cos(\omega t - 2m\theta) + \cos(\omega t + 2m\theta))$$

$$\times\left(\sum_{m=0,\,1,\,\cdots}(-2mG_{1m-}(J)\cos(\omega t - 2m\theta) + 2mG_{1m+}(J)\cos(\omega t + 2m\theta))\right)$$

Multiplying out and rewriting gives rise to terms of the form

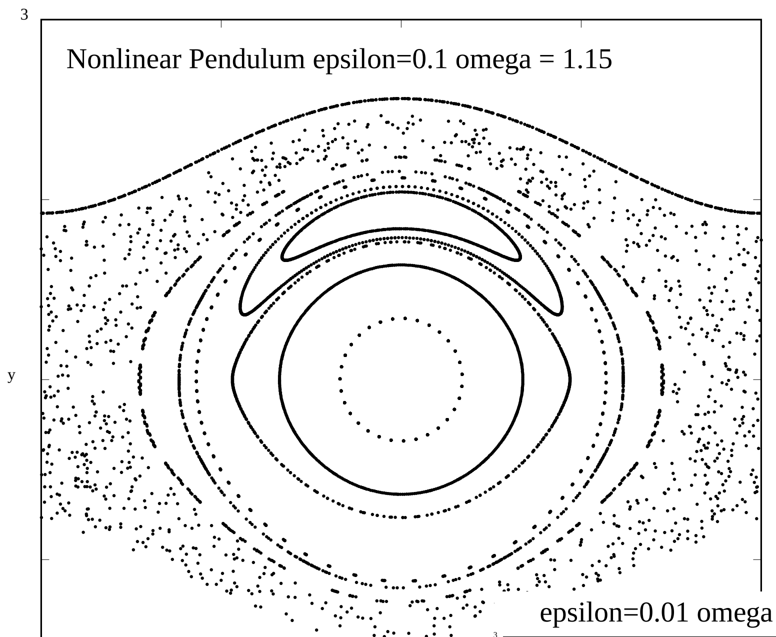
$$\cos(\omega t + 2\ell_1\theta)\cos(\omega t + 2\ell_2\theta) = \frac{1}{2}\left(\cos(2\omega t + 2(\ell_1 + \ell_2)\theta) + \cos(2(\ell_1 - \ell_2)\theta)\right)$$

The first term introduces new resonances at

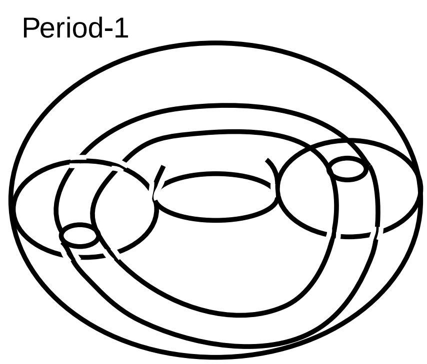
$$\frac{\omega(I)}{\omega} = \pm\frac{2}{2\ell} = \frac{1}{\ell} \qquad \text{these are the second order resonance conditions.}$$

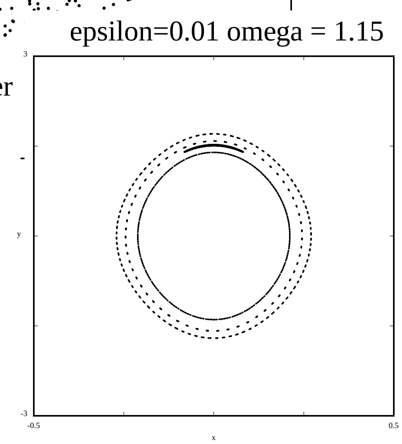
The second order resonances,occurring at $\frac{\omega(I)}{\omega} = \pm\frac{2}{2\ell} = \frac{1}{\ell}$, can be investigated in a similar manner, via a rotating frame coordinate $\psi = \theta - \frac{\omega t}{\ell}$.

They will be period-ℓ, which means all the odd periods are second order. Because they occur at order ϵ^2 their width is order ϵ (the width of the first order resonances is order $\sqrt{\epsilon}$). So you need higher values of ϵ to see them.



Note that the period-1's are much smaller for small epsilon because they are second order resonances.





There are two similar, but totally separate tori which make up the period-1 resonances.

Higher order resonances

Third order resonances occur when $\frac{\omega(I)}{\omega} = \pm\frac{3}{2\ell}$ and have width on the order of $\epsilon^{\frac{3}{2}}$. In general

$$n\text{th order resonances occur at} \quad \frac{\omega(I)}{\omega} = \pm\frac{n}{2\ell} \quad \text{and have width } \epsilon^{\frac{n}{2}}.$$

Despite the presence of resonances, for ϵ small, the overall behaviour can appear fairly regular away from the main separatrix. This is because most of the tori with $\frac{\omega(I)}{\omega}$ irrational persist and their existence limits the extent of the chaos. As ϵ increases the width of the resonances increases and so the integrable tori between them are destroyed. In fact this idea gives rise to a rather crude estimate for global chaos.

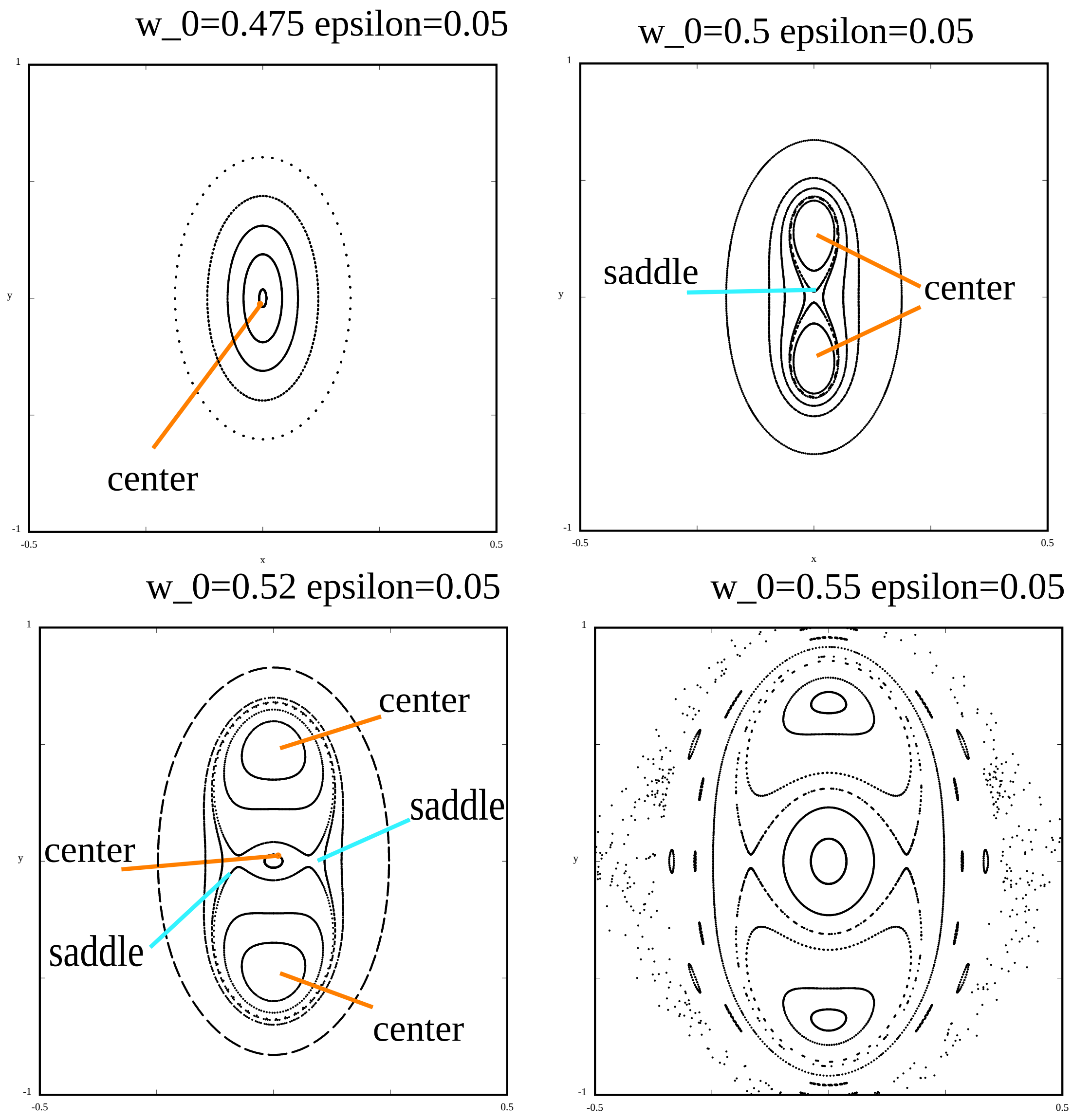
Resonances emerging from the origin.

The resonance condition; $\frac{\omega(I)}{\omega} = \frac{n}{2m}$, for an order n resonance, may not always have a solution. Since

$$\omega(I) \le \omega_0 \quad \text{there is some } I \text{ if and only if} \frac{n\omega}{2m} \le \omega_0.$$

For instance the period-2 resonance exists if $\omega_0 \ge \frac{\omega}{2}$. One approach to analysing a system is to fix $\omega = 1$ (you can always scale time). Then the period-2 resonance exists for $\omega_0 \ge \frac{1}{2}$, the period-1 exists for $\omega_0 \ge 1$etc.

It is quite interesting to see it emerge. Two bifurcations take place to the Poincare Map. The stable center at the origin undergoes a pitchfork bifurcation to an unstable saddle. Then the saddle undergoes a second pitchfork bifurcation to a stable center.



Perturbation Theory for 2 d.o.f. systems.

The method used for m d.o.f. systems is similar to that for $1\frac{1}{2}$ d.o.f. systems. We assume that the unperturbed Hamiltonian is integrable and so a function of the actions only. Then using a near identity generating function we try to remove all the θ dependent terms. So assume that

$$H(\theta_1, \theta_2, I_1, I_2) = H_0(I_1, I_2) + \epsilon H_1(\theta_1, \theta_2, I_1, I_2) + \epsilon^2 H_2 + \cdots$$

and look for a near identity generating function that transforms the sustem to action angle variables.

$$F_2(\theta_1, \theta_2, J_1, J_2) = J_1\theta_1 + J_2\theta_2 + \epsilon G(\theta_1, \theta_2, J_1, J_2)$$

where $G = G_1 + \epsilon G_2 + \cdots$. Then

$$I_i = J_i + \epsilon \frac{\partial G}{\partial \theta_i} \qquad \phi_i = \theta_i + \epsilon \frac{\partial G}{\partial J_i}$$

and the new Hamiltonian is simply the old Hamiltonian written as a function of the new variables:

$$K(\theta_1, \theta_2, J_1, J_2) = H_0(J_1, J_2) + \epsilon \left(\frac{\partial H_0}{\partial I_1}(J_1, J_2)\frac{\partial G_1}{\partial \theta_1} + \frac{\partial H_0}{\partial I_2}(J_1, J_2)\frac{\partial G_1}{\partial \theta_2} + H_1(\theta_i, J_i) \right)$$

$$+\frac{\epsilon^2}{2}\left(\frac{\partial^2 H_0}{\partial I_1^2}(J_1, J_2)\left(\frac{\partial G_1}{\partial \theta_1}\right)^2 + \frac{\partial^2 H_0}{\partial I_2^2}(J_1, J_2)\left(\frac{\partial G_1}{\partial \theta_2}\right)^2 + 2\frac{\partial^2 H_0}{\partial I_1 \partial I_2}(J_1, J_2)\frac{\partial G_1}{\partial \theta_1}\frac{\partial G_1}{\partial \theta_2} \right)$$

$$+\epsilon^2 \left(\frac{\partial H_1}{\partial I_1}\frac{\partial G_1}{\partial \theta_1} + \frac{\partial H_1}{\partial I_2}\frac{\partial G_1}{\partial \theta_2} + \frac{\partial H_0}{\partial I_1}\frac{\partial G_2}{\partial \theta_1} + \frac{\partial H_0}{\partial I_2}\frac{\partial G_2}{\partial \theta_2} + H_2 \right) + \cdots$$

Once again we use G_1 to remove any θ_i dependent terms at order ϵ, then G_2 to remove any θ_i dependent terms at order ϵ^2 etc. To do this systematically expand H_i as Fourier series in θ_i, then say if

$$H_1(\theta_i, I_i) = \sum_{n_1, n_2} V_{n_1,n_2}(I_i)\cos(n_1\theta_1 - n_2\theta_2)$$

we choose $G_1 = \sum_{n_1,n_2} G_{1,n_1,n_2}$ such that

$$\frac{\partial H_0}{\partial I_1}(J_1, J_2)\frac{\partial G_{1,n_1,n_2}}{\partial \theta_1} + \frac{\partial H_0}{\partial I_2}(J_1, J_2)\frac{\partial G_{1,n_1,n_2}}{\partial \theta_2} + V_{n_1,n_2}(J_i)\cos(n_1\theta_1 - n_2\theta_2) = 0$$

So let

$$G_{1,n_1,n_2} = -\frac{V_{n_1,n_2}(J_i)}{n_1\omega_1(J_i) - n_2\omega_2(J_i)}\sin(n_1\theta_1 - n_2\theta_2) \quad \text{where} \quad \omega_i(J_1, J_2) = \frac{\partial H_0}{\partial I_i}(J_1, J_2)$$

This is only possible if the denominator is nonzero.

The first order resonance condition is $\quad n_1\frac{\partial H_0}{\partial I_1}(J_1, J_2) - n_2\frac{\partial H_0}{\partial I_2}(J_1, J_2) = 0$

Or $n_1\omega_1(J_i) - n_2\omega_2(J_i) = 0$. If J_i are such that this is nonzero the term can be removed.

Take as an example the following Hamiltonian

$$H = I_1 + I_2 - I_1^2 - 3I_1I_2 + I_2^2 + \alpha I_1I_2\cos(2\theta_1 - 2\theta_2) + \beta I_1I_2^{\frac{3}{2}}\cos(2\theta_1 - 3\theta_2)$$

for α and β small. Then there will be first order resonances when

$$\frac{\partial H_0}{\partial I_1} = \frac{\partial H_0}{\partial I_2} \qquad \Rightarrow 1 - 2I_1 - 3I_2 = 1 - 3I_1 + 2I_2 \qquad \Rightarrow I_1 = 5I_2,$$

$$2\frac{\partial H_0}{\partial I_1} = 3\frac{\partial H_0}{\partial I_2} \qquad \Rightarrow 2(1 - 2I_1 - 3I_2) = 3(1 - 3I_1 + 2I_2) \qquad \Rightarrow 5I_1 = 1 + 12I_2$$

Investigaton of a general resonance.

Suppose that a near identity generating function has been used to remove all oscillating terms at order ϵ except one:

$$H = H_0(J_i) + \epsilon V_{n_1,n_2}(J_i)\cos(n_1\theta_1 - n_2\theta_2)$$

Then moving to rotating coordinates: $\psi_1 = \theta_1 - \dfrac{n_2}{n_1}\theta_2 \quad \psi_2 = \theta_2$ using the generating function $F_2(\theta_i,\, L_i) = L_1(\theta_1 - \frac{n_2}{n_1}\theta_2) + L_2\theta_2$, gives new momenta L_i;

$$J_1 = L_1 \qquad \text{and} \qquad J_2 = L_2 - \frac{n_2}{n_1}L_1$$

Then if we let $H_0(L_1,\, L_2 - \frac{n_2}{n_1}L_1) = \bar{H}_0(L_1,\, L_2)$ and $V(L_1,\, L_2 - \frac{n_2}{n_1}L_1) = \bar{V}(L_1,\, L_2)$ the new Hamiltonian is

$$H = \bar{H}_0(L_1,\, L_2) + \epsilon\bar{V}(L_1,\, L_2)\cos(n_1\psi_1)$$

Since the new Hamiltonian is not a function of ψ_2 the new momentum L_2 is a constant of the motion. Now at resonance $L_1 = J_{10}$ and L_2 is a constant, so that J_{10} is given by

$$n_1\omega_1(J_{10},\, L_2 - \frac{n_2}{n_1}J_{10}) - n_2\omega_2(J_{10},\, L_2 - \frac{n_2}{n_1}J_{10}) = 0 \quad \text{where} \quad \omega_i(J_1,\, J_2) = \frac{\partial H_0}{\partial J_i}$$

However it would be useful to rewrite this in terms of $\bar{H}$. Consider

$$\frac{\partial \bar{H}_0}{\partial L_1} = \frac{\partial H_0}{\partial J_1}\frac{\partial J_1}{\partial L_1} + \frac{\partial H_0}{\partial J_2}\frac{\partial J_2}{\partial L_1} = \frac{\partial H_0}{\partial J_1} + \frac{\partial H_0}{\partial J_2}\left(-\frac{n_2}{n_1}\right) = \omega_1 - \frac{n_2}{n_1}\omega_2$$

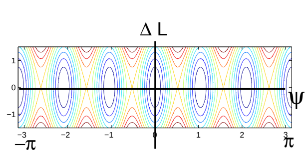
So at resonance $\dfrac{\partial \bar{H}_0}{\partial L_1}(J_{10},\, L_2) = 0$

Since we are interested in the behaviour near resonance let $L_1 = J_{10} + \sqrt{\epsilon}\Delta L_1$. Then useing Taylor series

$$H = \bar{H}_0(J_{10},\, L_2) + \sqrt{\epsilon}\frac{\partial \bar{H}_0}{\partial L_1}(J_{10},\, ,L_2)\Delta L + \frac{\epsilon}{2}\frac{\partial^2 \bar{H}_0}{\partial L_1^2}(J_{10},\, L_2)\Delta L_1^2 + \epsilon\bar{V}(J_{10},\, L_2)\cos(n_1\psi) + \cdots$$

which becomes $\qquad H = \bar{H}_0(J_{10},\, L_2) + \dfrac{\epsilon}{2}\dfrac{\partial^2 \bar{H}_0}{\partial L_1^2}(J_{10},\, L_2)\Delta L_1^2 + \epsilon\bar{V}(J_{10},\, L_2)\cos(n_1\psi) + \cdots$

which is the nonlinear pendulum with n_1 eyes!



A single resonance Hamiltonian is integrable, as there are two integrals of the motion H and L_2, but a second resonance is enough to destroy the integrability.

Consider

$$H = H_0(I_1,\, I_2) + \alpha I_1 I_2 \cos(2\theta_1 - 2\theta_2) + \beta I_1 I_2^{\frac{3}{2}} \cos(2\theta_1 - 3\theta_2)$$

the best we can do is stay away from the resonances, assuming that α and β are small. But this will be hard to do for all resonances to all orders.First order resonances occur at $\omega_1 - \omega_2 = 0$ and $2\omega_1 - 3\omega_2 = 0$, but second order will occur at $4\omega_1 - 5\omega_2 = 0$ and $\omega_2 = 0$, third order will occur at $6\omega_1 - 7\omega_2 = 0$, $6\omega_1 - 8\omega_2 = 0$ and $2\omega_1 - 4\omega_2 = 0$ etc. There are an infinite numer of resonance conditions. However the higher order resonances will have reduced width: $\alpha^{\frac{n}{2}}$, $\beta^{\frac{n}{2}}$, for order n. So it is just possible that we could stay away from all the resonances for α and β small. This is the subject of the KAM theorem

Canonical Perturbations via Lie transforms.

The evolution of a system under any Hamiltonian generates a continuous family of canonical transformations. Consider the Hamiltonian

$$W(r,\, \theta,\, p_r,\, p_\theta : \tau) = p_\theta \qquad \text{where the evolution is in the variable } \tau.$$

Since Hamilton's equations of motion are

$$\dot{r} = \frac{\partial W}{\partial p_r} = 0, \qquad \dot{\theta} = 1, \qquad \dot{p_r} = 0, \qquad \dot{p_\theta} = 0, \qquad \text{where} \quad \dot{f} = \frac{df}{d\tau}$$

$$\Rightarrow \qquad r = r', \qquad \theta = \theta' + \tau, \qquad p_r = p_r', \qquad p_\theta = p_\theta'.$$

The angular momentum is the generator of a rotation in θ.

Or consider the Hamiltonian

$$W(x,\, y,\, z,\, p_x,\, p_y,\, p_z) = xp_y - yp_x, \qquad \text{where}$$

$$\dot{x} = -y, \qquad \dot{y} = x, \qquad \dot{z} = 0, \qquad \dot{p_x} = -p_y, \qquad \dot{p_y} = p_x, \qquad \dot{p_z} = 0$$

This gives a rotation in x, y about the z axis and in p_x, p_y.

To use this suppose

$$H(x,\, y,\, p_x,\, p_y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{a}{2}(x-y)^2 + \frac{b}{2}(x+y)^2$$

then using the Hamiltonian W we can effect a rotation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} p_x' \\ p_y' \end{pmatrix}$$

Here the trick is to choose $\tau = \frac{\pi}{4}$ then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_x' \\ p_y' \end{pmatrix}$$

So that the final Hamiltonian has the form

$$H(x',\, y',\, p_x',\, p_y') = \frac{(p_x')^2}{2} + \frac{(p_y')^2}{2} + a(x')^2 + b(y')^2$$

which is in separated form and therefore integrable.

In **Canonical perturbation theory** the change in the canonical variables is also a continuous change. The variable of change is ϵ. If $\epsilon = 0$ then $I = J$ and $\theta = \phi$. As a function f evolves under ϵ via some Hamiltonian W

$$\frac{df}{d\epsilon} = \{f, W\} \equiv L_W f \quad \text{The Lie derivative operator of } f$$

Similarily

$$L_W^2 f = \{\{f, W\}, , W\} \quad \text{which is} \frac{d^2 f}{d\epsilon^2}$$

etc. Now take a Taylor series

$$f(\tau+\epsilon) = f(\tau)+\epsilon\frac{df}{d\epsilon}(\tau)+\frac{\epsilon^2}{2}\frac{d^2 f}{d\epsilon}(\tau)+\cdots = f(\tau)+\epsilon L_W f(\tau)+\frac{\epsilon^2}{2}L_W^2 f(\tau)+\cdots = e^{\epsilon L_W} f(\tau)$$

This means that W generates a caonical transform via the operator $e^{\epsilon L_W}$. The new variables are given by

$$H' = e^{\epsilon L_W} H, \quad q = e^{\epsilon L_W} q', \quad p = e^{\epsilon L_W} p'$$

Now suppose that

$$H = H_0 + \epsilon H_1 \qquad \text{with} \qquad H_0(I_i)$$

Then the evolved Hamiltonian is

$$H' = e^{\epsilon L_W} H = e^{\epsilon L_W}(H_0 + \epsilon H_1) = H_0 + \epsilon(L_W H_0 + H_1) + \epsilon^2 \left(\frac{L_W^2}{2} H_0 + L_W H_1\right) + \cdots$$

The attraction of this method is that it is easy to find a general expression for the perturbation at any order.

Now suppose we wish to remove the oscillating terms at order ϵ. Separate the oscillating and nonoscillating parts. So if the average of H_1 is $\langle H_1 \rangle$

$$\text{let} \quad H_1 = \langle H_1 \rangle + \bar{H}_1 \qquad \text{and choose } W \text{ so that} \quad L_W H_0 + \bar{H}_1 = 0$$

So if for instance $H_1 = V_0(I_1, I_2)+V_1(I_1, I_2)\cos(n_1\theta_1 - n_2\theta_2)$, for which $\langle H_1 \rangle = V_0(I_1, I_2)$, we choose W such that

$$L_W H_0 = \{H_0, W\} = \frac{\partial H_0}{\partial I_1}\frac{\partial W}{\partial \theta_1} + \frac{\partial H_0}{\partial I_2}\frac{\partial W}{\partial \theta_2} = -\bar{H}_1 = -V_1(I_1, I_2)\cos(n_1\theta_1 - n_2\theta_2).$$

To remove terms at all orders we require W itself to be a series in ϵ. The result is more complicated, however this is a good method for higher order calculations because the higher order terms can be generated systematically.