Section 1.6 The Kolmogorov, Arnold and Moser Theorem.

The KAM theorem concerns perturbed time independent Hamiltonians of any degree, say N, which are integrable for $\epsilon = 0$. So

$$H(\theta_i,\, I_i) = H_0(I_i) + \epsilon V(\theta_i,\, I_i) \quad \text{for} \quad i = 1...N$$

where $V(\theta_i,\, I_i)$ is represented by it's Fourier series;

$$V = \sum_{n_1}\sum_{n_2}\sum_{n_3}\cdots\sum_{n_N} V_{n_1,n_2...n_N}(I_i)e^{i(n_1\theta_1+\cdots n_N\theta_N)}, \qquad \text{for some} \quad V_{n_1,n_2...n_N}$$

There are two main conditions. The first is on the Fourier coefficients $V_{n_1,n_2...n_N}$ and the second is on the unperturbed Hamiltonian $H_0(I_i)$.

1 V must be a smooth function of the of the angles θ_i, that is it has sufficient partial derivatives with respect to θ_i, so that the Fourier coefficients $V_{n_1,n_2...n_N}$ decrease fairly rapidly with increasing n_i.

2 The Hessian of $H_0(I_i)$, that is

$$det\left(\frac{\partial^2 H_0}{\partial I_i \partial I_j}\right) \qquad \text{is nonzero.}$$

Then KAM showed that **the volume of phase space occupied by the resonances goes to zero as $\epsilon \to 0$.**

Also **if ϵ is small enough then on a torus** $\omega_i = \frac{\partial H_0}{\partial J_i}$ such that

$$|\sum_{i=1}^{N} n_i\omega_i| \geq \frac{K}{|\sum_{i=1}|n_i||^{\alpha}} \qquad \text{for all integers } n_i \text{ and for some constants } \alpha \text{ and } K$$

a perturbation theory will converge.

The proof involves Canonical perturbation theory, which may be carried out to all orders in ϵ by using Lie transform methods. Proving that the perturbation theory converges for certain irrational tori involves estimates on the width of the resonances that we know replace the rational tori.

Condition 1, which is a condition on the size of the fourier coefficients, ensures that the width, as we calculated them in the previous section, is small enough for all resonances to remain separated so that invariant tori can exist between them.

However the description of a resonance and it's width requires that the unperturbed Hamiltonian is sufficiently nonlinear and more. Condition 2, which implies that the resonances are **accidentally degenerate**, ensures that the unperturbed Hamiltonian is sufficiently nonlinear and more, such that the resonance conditions on the unperturbed Hamiltonian are isolated.

Accidentally degenerate or intrinsically degenerate.

If the unperturbed Hamiltonian is linear;

$$H_0(I_i) = \sum_i a_i I_i \quad \text{then} \quad \frac{\partial^2 H_0}{\partial I_i \partial I_j} = 0$$

so the Hessian of H_0 is zero. Then any resonance condition;

$$\sum_i n_i \frac{\partial H_0}{\partial I_i} = \sum_i n_i a_i$$

gives no condition on the unperturbed actions. It is either satisfied for all I_i, or never satisfied. Such a resonance is said to be **intrinsically degenerate.** The resonances are not isolated. So the KAM theorem does not apply to a perturbed linear Hamiltonian, such as the Henon Hiles system. But there are also some nonlinear Hamiltonians which have intrinsically degenerate resonances.

In all the examples nonlinear systems we have looked at so far we have assumed that the unperturbed Hamiltonian had nonzero Hessian. But take the following example where this is not the case.

$$H(\theta_1, \theta_2, I_1, I_2) = H_0(I_1 + I_2) + \epsilon V(I_1, I_2)\cos(\theta_1 - \theta_2)$$

The resonance condition is

$$\frac{\partial H_0}{\partial I_1} - \frac{\partial H_0}{\partial I_2} = 0, \qquad \text{which is always satisfied!}$$

So even if H_0 is nonlinear a resonance condition may not give a condition on the unperturbed actions, in which case the resonance is intrinsically degenerate. To ensure that this is not the case, that is that the resonance is **accidentally degenerate**, we require that the Hessian of H_0 be nonzero. This then implies that the resonances are isolated. To see this consider the following Hamiltonian with a relatively simple unperturbed part, whose Hessian is nonzero $(Hessian(H_0) = 1)$

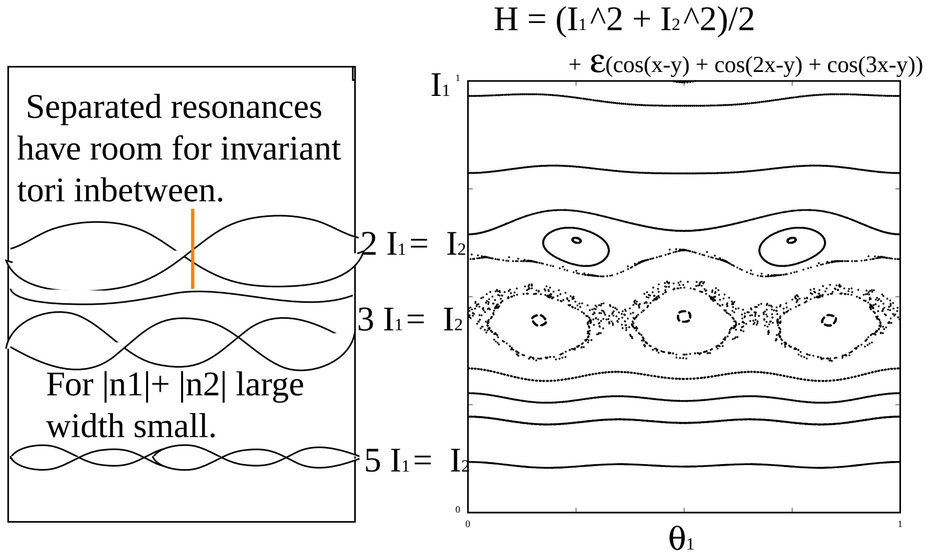
$$H = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \sum_{n_i} V_{n_i}(I_i) e^{i(n_1\theta_1 + n_2\theta_2)}$$

then $\omega_1 = I_1$ and $\omega_2 = I_2$. The resonance condition becomes

$$n_1\omega_1 + n_2\omega_2 = 0 \qquad \Rightarrow \qquad n_1 I_1 + n_2 I_2 = 0$$

So assuming that $V_{n_1,n_2} \neq 0$ there are resonances at $I_1 = -\frac{n_2}{n_1} I_2$ which if we consider the poincare Map $\theta_2 = 0$ will have n_1 eyes.

Here is a Poincare map of a Hamiltonian System similar to the one above with a couple of resonances. For ϵ smaller their widths would be reduced.



The volume taken up by the resonances, of which there are an infinite number as the rational are dense in the reals, is never less limited enough to allow for invariant tori. At first this seems unlikely. But consider taking away an interval of length $\dfrac{2\epsilon}{n^3}$ about each rational $\frac{m}{n}$. Then the length occupied by the resonances is

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{2\epsilon}{n^3} = 2\epsilon\sum_{n=1}^{\infty}\frac{1}{n^2} = \epsilon\frac{\pi^2}{3} < 1 \quad \text{and} \to 0 \quad \text{as} \quad \epsilon \to 0$$

So for ϵ small only a small fraction of the volume is taken up with resonance zones. The rest comprises invariant tori.

But do all the tori with $\frac{\omega_1}{\omega_2}$ irrational survive? No.

The KAM theorem says that if ϵ is small enough then on a torus $\omega_i = \frac{\partial H_0}{\partial J_i}$ such that

$$|\sum_{i=1}^{N} n_i\omega_i| \geq \frac{K}{|\sum_{i=1}|n_i||^{\alpha}} \qquad \text{for all integers } n_i \text{ and for some constants } \alpha \text{ and } K$$

a perturbation theory will converge.

For a two degree of freedom system this means that

$$|n_1\omega_1 + n_2\omega_2| \geq \frac{K}{||n_1| + |n_2||^{\alpha}} \qquad \text{for all integers } n_1 \text{ and } n_2 \text{ and some constants } \alpha \text{ and } K.$$

This is a **Diophantine condition**. We will look at these in some depth in the section on the Standard Map. In vague terms it means that the tori that survive are those sufficiently far away from any rational. Continued fractions are the way to understand Diophantine conditions.