## Section 1.7 Twist Maps.

Maps may arise naturally as approximations to kicked systems or they may be constructed as Poicare Maps of continuous systems.

Fermi acceleration, which models cosmic ray acceleration where charged particles are accelerated by collisions with magnetic field structures, is modeled by a map. The resulting dynamics is that of a ball bouncing between a fixed and oscillating wall. The simplest version of this allows the oscillating wall to impart momentum to the ball, according to the balls velocity, without the wall changing it's position in space.

So let $u_n$ be the normalized velocity and $\psi_n$ be the phase of the moving ball, just before the $n$th collision of the ball with the fixed wall. then

$$u_{n+1} = |u_n + F(\psi_n)| \qquad \psi_{n+1} = \psi_n + \frac{M}{u_{n+1}}$$

where $F(\psi_n)$ is the velocity impulse given to the ball.

There are a number of models for the velocity impulse,

$$F(\psi) = \psi - \tfrac{1}{2} \qquad \text{sawtooth wall.}$$

$$F(\psi) = \sin(\psi) \qquad \text{sinusoidal momentum transfer}$$

In very simple cases we can construct the map from a continuous system. Consider the integrable Hamiltonian system

$$H = H_0(I_1,\, I_2) \qquad \Rightarrow \dot{\theta}_i = \frac{\partial H_0}{\partial I_i} = \omega_i \qquad \Rightarrow \theta_i = \omega_i(t - t_0)$$

Take a Poincare map with surface of section $\theta_2 =$ constant, that is

$$\theta_2 = \theta_{20} \qquad \Rightarrow t_n - t_0 = \frac{\theta_{20} + 2n\pi}{\omega_2}$$

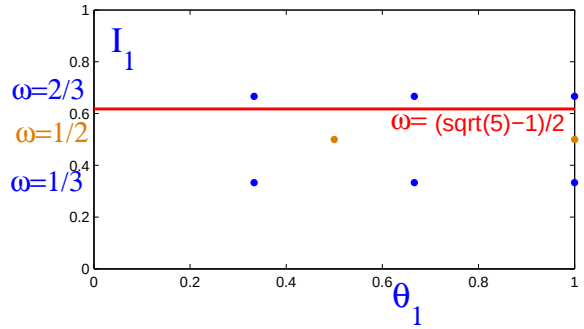
which is equivalent to taking a strobe map of the $(\theta_1,\, I_1)$ space.

$$\theta_1(t_{n+1}) = \omega_1 \left( \frac{\theta_{20} + 2(n+1)\pi}{\omega_2} \right) = \frac{\omega_1}{\omega_2} 2\pi + \theta_1(t_n)$$

So if $\theta_n = \theta_1(t_n)$

$$\theta_{n+1} = \frac{\omega_1}{\omega_2} 2\pi + \theta_n \qquad I_{n+1} = I_n$$

The ratio of frequencies $\frac{\omega_1}{\omega_2} = \omega(I_i)$ is called the **winding number.**




If $\omega(I_i)$ is a rational
then the solution is periodic.
The map is just a finite set of points.
If $\omega(I_i)$ is irrational
the iterates cover a line.

As the flow evolves it does so canonically. So that

$\{\theta_{n+1},\, I_{n+1}\}_{\{\theta_n,\, I_n\}} = 1$ via an $F_2(\theta_n,\, I_{n+1})$ generating function. $\left(\theta_{n+1} = \dfrac{\partial F_2}{\partial I_{n+1}},\, I_n = \dfrac{\partial F_2}{\partial \theta_n}\right)$

The first part of the generating function generates the identity.

$$F_2(\theta_n,\, I_{n+1}) = \theta_n I_{n+1} + 2\pi A(I_{n+1}) \quad \text{where} \quad \frac{dA(I_{n+1})}{dI_{n+1}} = \omega(I_{n+1})$$

Perturbed twist maps that evolve canonically have a specific form. Consider a general perturbation of the simple twist map:

$$\begin{aligned} \theta_{n+1} &= \theta_n + 2\pi\omega(I_{n+1}) + \epsilon g(\theta_n,\, I_{n+1}) \\ I_{n+1} &= I_n + \epsilon f(\theta_n,\, I_{n+1}) \end{aligned}$$

For this to be generated by an $F_2(\theta_n,\, I_{n+1})$ generating function we require

$$\frac{\partial F_2}{\partial I_{n+1}} = \theta_n + 2\pi\omega(I_{n+1}) + \epsilon g(\theta_n,\, I_{n+1}) \quad \text{and} \quad \frac{\partial F_2}{\partial \theta_n} = I_n - \epsilon f(\theta_n,\, I_{n+1})$$

This is satisfied if

$$F_2(\theta_n,\, I_{n+1}) = \theta_n I_{n+1} + 2\pi A(I_{n+1}) + \epsilon G(\theta_n,\, I_{n+1}) \quad \text{where} \quad f = -\frac{\partial G}{\partial \theta_n} \quad \text{and} \quad g = \frac{\partial G}{\partial I_{n+1}}$$

So we require

$$\frac{\partial f}{\partial I_{n+1}} + \frac{\partial g}{\partial \theta_n} = 0$$

Even quite simple perturbations of this, perturbed twist maps, often have chaotic solutions. Even the radial twist map

$$\begin{aligned} \theta_{n+1} &= \theta_n + 2\pi\omega(I_{n+1}) \\ I_{n+1} &= I_n + \epsilon f(\theta_n) \end{aligned}$$

or it's *simple* example the **Standard Map**

$$\begin{aligned} \theta_{n+1} &= \theta_n + I_{n+1} \pmod{2\pi} \\ I_{n+1} &= I_n + K\sin\theta_n \end{aligned}$$

displays a whole range of resonance behavior. The Standard Map has received a lot of attention, partly because it provides an approximation to various other maps, such as the Fermi map and the separatrix map.

**Critical points of Maps.**

For a given two dimensional map

$$\begin{pmatrix} \theta_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{pmatrix} \theta_{n+1}(\theta_n,\, I_n) \\ I_{n+1}(\theta_n,\, I_n) \end{pmatrix} \quad (\theta^*,\, I^*) \text{ is a critical point iff } \begin{array}{c} \theta^* = \theta_{n+1}(\theta^*,\, I^*) \text{ AND} \\ I^* = I_{n+1}(\theta^*,\, I^*) \end{array}$$

**The Tangent map or the linearized system.**

The linearization about critical points of a map is often called the tangent map.

$$\begin{pmatrix} \theta_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_{n+1}}{\partial \theta_n} & \frac{\partial \theta_{n+1}}{\partial I_n} \\ \frac{\partial I_{n+1}}{\partial \theta_n} & \frac{\partial I_{n+1}}{\partial I_n} \end{pmatrix}_{(\theta^*, I^*)} \begin{pmatrix} \theta_n \\ I_n \end{pmatrix} = T \begin{pmatrix} \theta_n \\ I_n \end{pmatrix}$$

Since the map is a canonical transformation $det(T) = 1$.

Now let $t = trace(T)$, then the eigenvalues of $T$ are given by

$$\lambda = \frac{t}{2} \pm \sqrt{\frac{t^2}{4} - 1} \quad \Rightarrow \quad \begin{array}{l} \lambda \text{ is complex if } \; -2 < t < 2 \\ \lambda \text{ is real if } \; t \le -2 \;\text{ or }\; t \ge 2 \end{array}$$

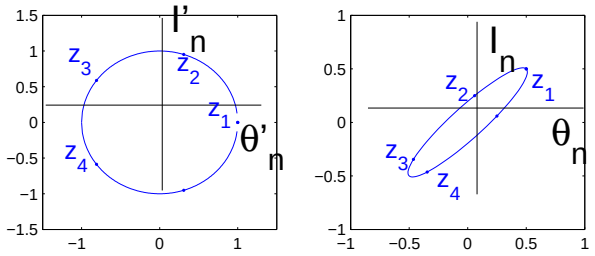
**If $\lambda$ is complex** we can transform to normal form

$$\begin{pmatrix} \theta'_{n+1} \\ I'_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \theta'_n \\ I'_n \end{pmatrix} \quad \text{where} \quad \begin{array}{l} \alpha = \frac{t}{2}, \;\; \beta = \sqrt{\frac{t^2}{4} - 1} \\ \text{so that} \;\; \alpha^2 + \beta^2 = 1 \end{array}$$

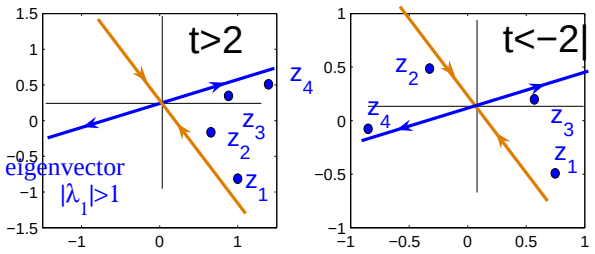
(The transform matrix $P$ such that $\begin{pmatrix} \theta'_n \\ I'_n \end{pmatrix} = P \begin{pmatrix} \theta_n \\ I_n \end{pmatrix}$ is $P = (Im(\mathbf{e}) Re(\mathbf{e}))$, for eigenvector $\mathbf{e}$.)

In normal form variables the linearized map is simply a rotation. We can let $\alpha = \cos\phi$ and $\beta = \sin\phi$ because $\alpha^2 + \beta^2 = 1$, for some $\phi$. So the solutions lie on circles.

Transforming back the solutions in $(\theta_n, I_n)$ space lie on ellipses.




**If $\lambda$ is real** $t < -2$ or $t > 2$ then the two eigenvalues $\lambda_1$ and $\lambda_2$ have product 1. So $\lambda_2 = \dfrac{1}{\lambda_1}$. This leads to saddle like behavior, if $|\lambda_1| > 1$, then $|\lambda_2| < 1$.




If $t = 2$ the map is the identity.

If $t = -2$ all points are period-2.

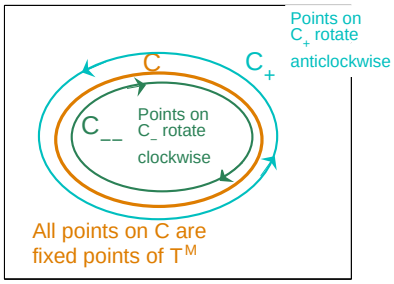
**Birkhoff's Fixed Point Theorem**

Take an area preserving map and an orbit with a rational winding number.

So that $\omega = \dfrac{N}{M}$. Then if

$$\begin{pmatrix} \theta_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{pmatrix} \theta_n + 2\pi\omega \\ I_n \end{pmatrix} = T \begin{pmatrix} \theta_n \\ I_n \end{pmatrix} \quad \Rightarrow \quad T^M \begin{pmatrix} \theta_n \\ I_n \end{pmatrix} = \begin{pmatrix} \theta_n \\ I_n \end{pmatrix}$$

That is all points on the orbit $C$ with $\omega = \frac{N}{M}$, are fixed points (critical points) of the map $T^M$.




Now suppose that $\omega < \dfrac{N}{M}$ on $C_-$, just inside $C$. Then

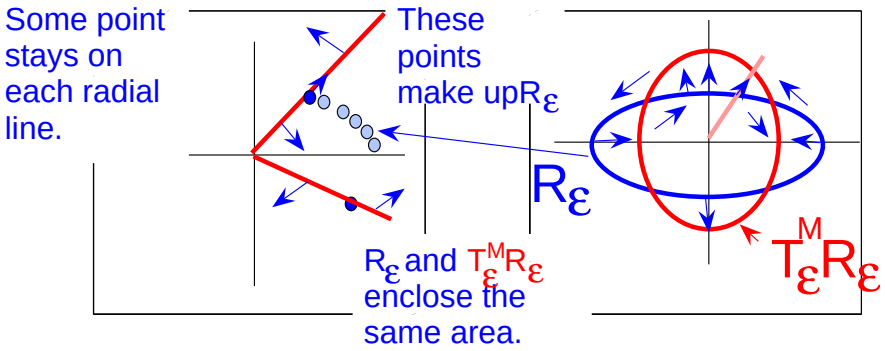
$$T^M \begin{pmatrix} \theta_n \\ I_n \end{pmatrix} = \begin{pmatrix} \theta_n + 2M\pi\omega \\ I_n \end{pmatrix} \quad \text{where } 2M\pi\omega < 2\pi N$$

so that points on $C_-$ rotate clockwise.

Similarly suppose that $\omega > \dfrac{N}{M}$ on $C_+$, so that points on $C_+$ rotate anticlockwise.

Now perturb the map to $T_\epsilon$

The relative twists will not be changed by $T_\epsilon^M$. Consider the points on a radial line, there will be outer points that rotate clockwise and inner points that rotate anticlockwise. So there must, by continuity, be a point on each radial line that does not rotate. That is it stays on the radial line. So under the map $T_\epsilon^M$ it moves either radially outwards or radially inwards.




Since this is true for all radial lines the points that do not rotate form a closed curve $R_\epsilon$.

Now consider the two curves $R_\epsilon$ and it's iterate $T_\epsilon^M R_\epsilon$. By area preservation they must cross in an even number of points.

Say $x^{(0)}$ is a fixed point of $T_\epsilon^M$, then so is $x^{(1)} = T_\epsilon x^{(0)}$, since

$$T_\epsilon^M x^{(0)} = x^{(0)} \quad \Rightarrow \quad T_\epsilon^M x^{(1)} = T_\epsilon^{M+1} x^{(0)} = T_\epsilon T_\epsilon^M x^{(0)} = T_\epsilon x^{(0)} = x^{(1)}$$

Similarly $x^{(m)}$ is a fixed point for $m = 0, 1, ...(m-1)$.

So there are $2kM$ fixed points of $T_\epsilon^M$.

Further the direction of the flow implies that their type alternates between center and saddle.

**Moser's KAM Theorem for Twist Maps**

Suppose a twist map is $j$ times differentiable and that

$$\int_0^{2\pi} f(I, \theta) d\theta = 0$$

so that the map doesn't simply shift points vertically.

**Then if**

$$|f|_j + |g|_j \quad \text{is sufficiently small} \quad (|f|_j = \sup_{m+n \le j} |\frac{\partial^{m+n} f}{\partial x^m \partial y^n}|)$$

**all rotationally invariant circles with winding number $\omega$ satisfying**

$$|m\omega - n| > K m^{-\alpha} \quad \forall n,\, m \in Z/0 \quad \text{and some} \quad \alpha \ge 1$$

**persist.**

The Diophantine condition implies that $\omega$ is poorly approximated by a rational. This condition is based on representing irrational numbers by continued fractions.