Section 1.8 The Standard Map.

The Standard Map, which has been analyzed extensively, has played an important role in classical and quantum chaos. It can be derived directly from a periodically kicked system; the kicked rotor. The Hamiltonian is

$$H(\theta,\, I) = \frac{I^2}{2} - K\cos\theta \sum_{n=-\infty}^{\infty} \delta(t-n)$$

Between kicks

$$H = \frac{I^2}{2} \qquad \Rightarrow I \text{ is a constant of the motion.} \Rightarrow \dot{\theta} = I \qquad \Rightarrow \qquad \theta = It + \theta_0.$$

So if $\Delta t = 1$ and $(\theta_n,\, I_n) = (\theta(t=n),\, I(t=n))$

$$\begin{aligned} \theta_{n+1} &= \theta_n + I_{n+1} \quad (\mathrm{mod}(2\pi)) \\ I_{n+1} &= I_n - K\sin\theta_n \end{aligned}$$

The Standard Map can also be derived as an approximate map for the solutions to the nonlinear pendulum near the separatrix. It has played an important part in understanding the chaos near separatrices.

If $K = 0$ then $I_{n+1} = I_n$ and the θ rotates by an amount I at each iteration. So that, like the unperturbed twist map

If $I = 2n\pi \;\;\Rightarrow\;\; \theta$ is fixed.
If $I = (2n-1)\pi \;\;\Rightarrow\;\; \theta$ is a line of period-2 points
If $I = \frac{2m\pi}{n} \Rightarrow \;\; \theta$ is a line of period-n points
If $\frac{I}{\pi}$ is irrational the iterates of the map slowly cover the line$I =$ constant.

The Standard Map may be thought of a perturbation, proportional to K, of the integrable twist map where $K = 0$.

Period-1, or fixed points, of the map can be found exactly.

$$\begin{aligned} \theta &= \theta + I\ (\mathrm{mod}(2\pi)) \qquad & \theta &= n\pi \quad \text{for} \quad n = 0 \pm 1 \\ I &= I - K\sin\theta & I &= 2m\pi \end{aligned}$$

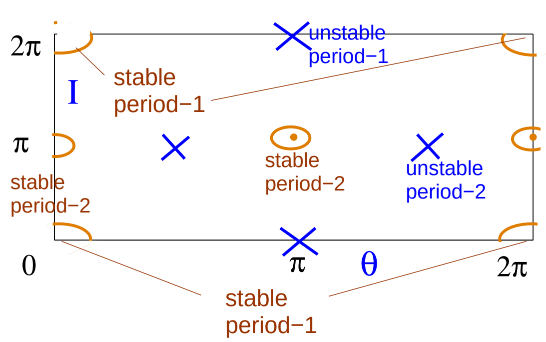
The linearized matrix is

$$T = \begin{pmatrix} 1 - K\cos\theta & 1 \\ -K\cos\theta & 1 \end{pmatrix} \qquad \text{trace} = 2 - K\cos\theta = 2 - K(-1)^n$$

So the period-1 points are stable if $-2 < 2 - K(-1)^n < 2$.

If n is odd and $K > 0$ then $2 - K(-1)^n > 2$, so they are always unstable.

If n is even they are stable(centers) for $K < 4$ and unstable for $K > 4$.



One **period-2 orbit of the map** $\mathbf{x}_0 = (\pi, 0)$, $\mathbf{x}_1 = (\pi, \pi)$ can be found exactly and its stability is straightforward to calculate. (There is a second whose position depends on K.)

Since a period-2 orbit of a map $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$ is a critical point of the map $\mathbf{f} \circ \mathbf{f}$ it is a stable period-2 orbit if it is a stable critical point of $\mathbf{f} \circ \mathbf{f}$.

Now the linearized matrix in general is the Jacobian matrix or $\mathbf{Df}$ and for a map $\mathbf{f} \circ \mathbf{f}$ it is $\mathbf{Df} \circ \mathbf{f}$ evaluated at the critical point. However using the chain rule one can show that this is $\mathbf{Df}(\mathbf{x}_1)\mathbf{Df}(\mathbf{x}_0)$.

So the stability of the period-2 orbit $(\pi, 0)$, (π, π), is determined from the product of the linearized matrix at each point of the orbit.

$$T(\pi, \pi)T(\pi, 0) = \begin{pmatrix} 1 - K\cos\pi & 1 \\ -K\cos\pi & 1 \end{pmatrix} \begin{pmatrix} 1 - K\cos 0 & 1 \\ -K\cos 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - K^2 - K & 2 + K \\ -K^2 & 1 + K \end{pmatrix}$$

The trace of the matrix is $2 - K^2$, so that this period-2 orbit is stable if $K^2 < 4 \Rightarrow K < 2$. In between this period-2 orbit is a period-2 orbit which is always unstable.

To obtain a crude estimate of the width of the resonances we can use the representation of the Standard map in Hamiltonian form.

$$H(\theta, I) = \frac{I^2}{2} - K\cos\theta \sum_{n=-\infty}^{\infty} \delta(t - n)$$

and expand the delta function as a cosine series:

$$\delta(n) = \sum_{m=-\infty}^{\infty} \delta(n - m) = 1 + 2\sum_{q=1}^{\infty} \cos(2\pi q n)$$

then

$$H(\theta, I) = \frac{I^2}{2} - K\cos\theta - 2K\cos\theta \sum_{q=1}^{\infty} \cos(2\pi q n)$$

$$H(\theta, I) = \frac{I^2}{2} - K\cos\theta - K\sum_{q=1}^{\infty} (\cos(\theta + 2\pi q n) + \cos(\theta - 2\pi q n))$$

where n behaves like time if we assume that $\frac{d\theta}{dn} \ll 1$.

Now there are first order resonances at

$$\frac{\partial H_0}{\partial I} \pm 2\pi q = 0 \qquad \Rightarrow I = 2\pi q$$

with width $\Delta I = 4\sqrt{K}$. Surprisingly large! For $K = 0.1$ the width is $\Delta I = 1.264$.

Resonance overlap.

A very crude estimate of the onset of global chaos can be found by considering when the resonances overlap. For instance the two period-1 resonances overlap when $\Delta I = 2\pi$. This implies that $4\sqrt{K} = 2\pi \Rightarrow K = \left(\frac{\pi}{2}\right)^2 \approx 2.47$. This is in fact a wild overestimate.

A better estimate can be obtained by considering overlap between the period-1's and the period-2's. First remove all the first order resonances via a near identity generating function.

$$F_2(\theta,\, J) = J\theta + Kf(\theta,\, J,\, n) \qquad I = J + K\frac{\partial f}{\partial \theta},\ \psi = \theta + K\frac{\partial f}{\partial J}$$

Then the new Hamiltonian is

$$K = H + \frac{\partial F_2}{\partial n} = \frac{\left(J + K\frac{\partial f}{\partial \theta}\right)^2}{2} + K\cos\theta + K\sum_{q=1}^{\infty}(\cos(\theta + 2\pi qn) + \cos(\theta - 2\pi qn)) + K\frac{\partial f}{\partial n}$$

So choose f such that

$$f(\theta,\, J,\, n) = f_0(J)\sin\theta + \sum_{q=1}^{\infty}(f_q(J)\sin(\theta + 2\pi qn) + f_{-q}(J)\sin(\theta - 2\pi qn))$$

where

$$f_0(J) = \frac{1}{J},\ f_q(J) = \frac{1}{J + 2\pi q}$$

This removes all the first order resonances and introduces the following terms at second order

$$\frac{K^2}{2}\left(\frac{\partial f}{\partial \theta}\right)^2 = \frac{K^2}{4}\sum_{\ell,m=-\infty} f_\ell(J)f_m(J)\left(\cos(2\theta + 2\pi(\ell+m)n) + \cos(2\pi(\ell - m)n)\right)$$

Now the period-2 at $I = \pi$ arises as a consequence of the $\cos(2\theta - 2\pi n)$ term. This coefficient

$$\frac{K^2}{4}\sum_{\ell+m=-1}^{\infty} f_\ell(J)f_m(J) = \frac{K^2}{4}\sum_{m=-\infty}^{\infty} f_m f_{-1-m}$$

Now $f_m(J)\frac{1}{J+2\pi m}$ so that $f_m f_{-1-m} = \frac{1}{(J+2\pi m)(J-2\pi(m+1))}$ Evaluated at $J=\pi$ gives

$$f_m(\pi)f_{-1-m}(\pi) = -\frac{1}{\pi^2(2m+1)^2}$$

$$\frac{K^2}{4}\sum_{m=-\infty}^{\infty} f_m f_{-1-m} = \frac{K^2}{4}\frac{1}{2} = \frac{K^2}{8}$$

So that $\Delta J = 4\sqrt{\frac{K^2}{8}} = K\sqrt{2}$. This means that overlap between the period-1's and period-2's occurs if

$$2\sqrt{K} + \sqrt{2}K = \pi \quad \Rightarrow \quad K \approx 1.26$$

still an over estimate, but much closer to the value $K \approx 1$ suggested by our numerical simulations.

Involutions of the Standard Map and finding Periodic orbits.

If a map I_1 is an involution then I_1^2 is the identity. The Standard Map is a product of two involutions, I_1 and I_2.

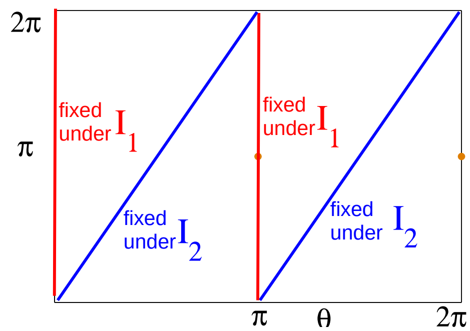
$$\begin{pmatrix} \theta_{n+1} \\ I_{n+1} \end{pmatrix} = I_2 I_1 \begin{pmatrix} \theta_n \\ I_n \end{pmatrix} \quad \text{where}$$

$$I_1 \begin{pmatrix} \theta_n \\ I_n \end{pmatrix} = \begin{pmatrix} -\theta_n \\ I_n - K\sin\theta_n \end{pmatrix} \quad I_2 \begin{pmatrix} \theta_n \\ I_n \end{pmatrix} = \begin{pmatrix} I_n - \theta_n \\ I_n \end{pmatrix}$$

Both I_1^2 and I_2^2 are the identity.

The presence of these involutions implies certain symmetries of the map which simplifies locating the period-m cycles.

For instance I_1 and I_2 have two fixed lines.



I_1 is fixed on $\theta_n = 0$ and on $\theta_n = \pi$.

I_2 is fixed on $\theta_n = \frac{I_n}{2}$ and on $\theta_n = \frac{I_n + 2\pi}{2}$.

You can prove that of the $2M$ symmetric cycles existing for $\omega = \frac{N}{M}$ one iterate lies on each of these lines.

This reduces the problem of finding period $2M$ symmetric cycles to a one dimensional search on the symmetry lines of the involutions. So the positions of the symmetric periodic orbits can be found numerically. To find any existing tori we need to understand continued fractions.

Rational Approximates to Irrationals and Continued Fractions.

Any irrational can be represented uniquely by a continued fraction

$$w \equiv a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}}} \equiv [a_0, a_1, a_2, a_3, ...]$$

where the a_i are positive integers ($a_i \geq 1$) for $i \geq 1$ and a_0 is a non negative integer($a_0 \geq 0$).

Then a_0 is the integer part of w. Now let

$$w_1 = \frac{1}{w - a_0} \quad \text{then} \quad w_1 = a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}}$$

so that a_1 is the integer part of w_1. Similarly define w_n recursively:

$$w_n = \frac{1}{w_{n-1} - a_{n-1}} \quad \text{then} \quad w_n = a_n + \cfrac{1}{a_{n+1} + \frac{1}{a_{n+2} + \frac{1}{\cdots}}}$$

so that a_n is the integer part of w_n.

If all the a_n's are nonzero w is an irrational. Rationals are obtained by truncating the series, typically written as $[a_0, a_1, a_2, a_3, ..., \infty]$. There is some non uniqueness in the representation of the rationals. For instance

$$\frac{1}{2} = [0,\ 1,\ 1,\ \infty] = 0 + \frac{1}{1+\frac{1}{1+0}} = 0 + \frac{1}{2+0} = [0,\ 2,\ \infty]$$

$$\frac{2}{3} = [0,\ 1,\ 1,\ 1,\ \infty], \qquad \frac{3}{5} = [0,\ 1,\ 1,\ 1,\ 1,\ \infty] \quad \text{or} \quad \frac{13}{30} = [0,\ 2,\ 3,\ 4,\ \infty]$$

$$\text{Or the irrational} \qquad \pi = [3,\ 7,\ 15,\ 293,\ ...]$$

By terminating the infinite sequence at say a_i we obtain the **Rational Approximates** to an irrational.

$$\frac{N_i}{M_i} = [a_0, a_1, a_2, a_3, ...,\ a_i,\ \infty]$$

These form a unique sequence of fractions converging to w.

The Rational Approximates are also the *best* approximation in the following sense:

$$|w - \frac{N}{M}| > |w - \frac{N_i}{M_i}| \quad \text{for all} \quad \frac{N}{M} \quad \text{with} \quad M < M_{i+1}$$

So for instance

$$\frac{\sqrt{5}-1}{2} = [0,\ 1,\ 1,\ 1,\ 1,\ ...] \approx 0.618034$$

has rational approximates $\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, ...$ but

$$|\frac{\sqrt{5}-1}{2} - \frac{3}{5}| = 0.018034 < \quad |\frac{\sqrt{5}-1}{2} - \frac{N}{6}| \quad \text{for any } N. \quad \text{and} \quad < |\frac{\sqrt{5}-1}{2} - \frac{N}{7}| \quad \text{for any } N$$

Now consider the **Diophantine Condition** in Moser's version of the KAM theorem for maps.

$$|m\omega - n| > \frac{K}{m^\alpha} \quad \forall m,\ n \in Z \qquad \Rightarrow |\omega - \frac{n}{m}| > \frac{K}{m^{\alpha+1}} \quad \forall m,\ n \in Z$$

It will be satisfied for some ω if it is satisfied for all the rational approximates to ω.

$$|\omega - \frac{N_i}{M_i}| > \frac{K}{M_i^{\alpha+1}}$$

By considering the fact that

$$\omega - \frac{N_0}{M_0} = \omega - a_0 = \frac{1}{a_1 + \frac{1}{a_2+\frac{1}{a_3+\frac{1}{\cdots}}}} \quad \text{is small if } a_1 \text{ is large}$$

and in general

$$\omega_n - a_n = \frac{1}{a_{n+1} + \frac{1}{a_{n+2}+\frac{1}{\cdots}}} \quad \text{is small if } a_{n+1} \text{ is large}$$

you can show that **Diophantine Conditions** are satisfied by those irrationals with their a_i's (at least) bounded above.

Further the **Noble Numbers**, which have continued fraction expansions that end in 1's

$$w = [a_0, a_1, a_2, a_3, ..., 1, 1, 1, ...] \quad \text{satisfy a } \textbf{Diophantine Condition} \text{ with } \alpha = 1.$$

The irrational worst approximated by rationals is the Golden mean

$$w = \frac{\sqrt{5}-1}{2} = [0, 1, 1, ...] \quad w^2 + w - 1 = 0 \Rightarrow \frac{1}{w} = \frac{1}{\frac{1}{w} - 1}$$

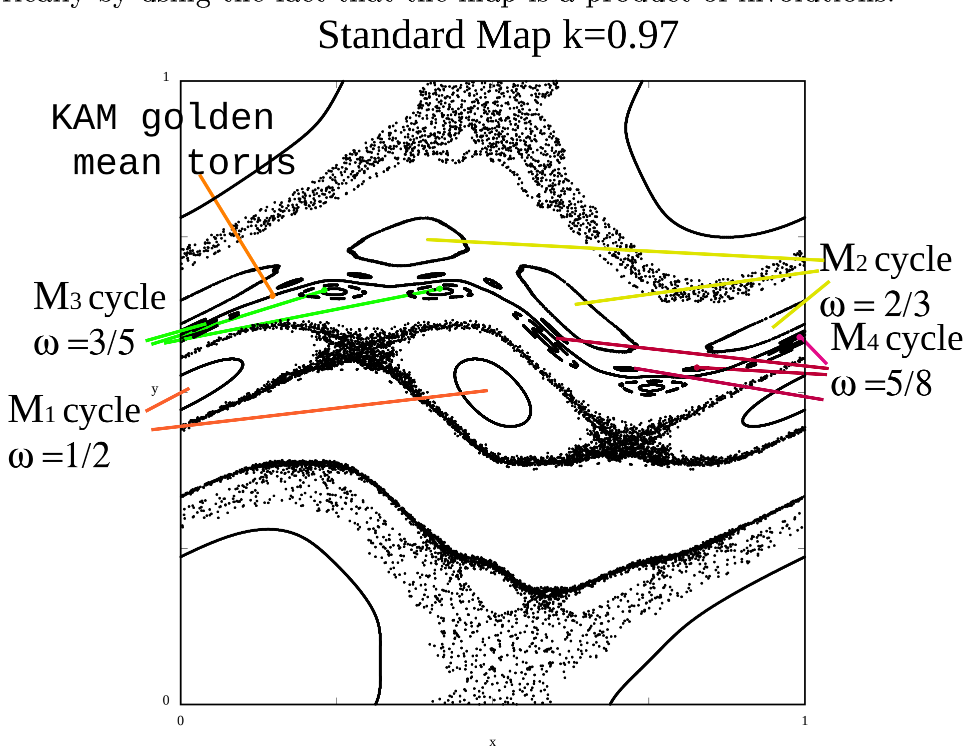
This means that the torus that is most stable to perturbations, according to the KAM Theorem, is that with winding number $\omega = \frac{\sqrt{5}-1}{2}$.

To locate the torus with $\omega = \frac{\sqrt{5}-1}{2}$ we use it's rational approximates:

$$\frac{1}{2} = [0, 1, 1, \infty], \; \frac{2}{3} = [0, 1, 1, 1, \infty], \; \frac{3}{5} = [0, 1, 1, 1, 1, \infty], \; \frac{5}{8}, \; \frac{8}{13} \quad \text{ratios of the Fibonacci numbers.}$$

and look for the corresponding M_i cycles.

Recall that the actual position of the M_i cycles in the standard map can be found numerically by using the fact that the map is a product of involutions.



The invariant tori are important because if they exist they bound the motion. For K small, where a large number of invariant tori exist, the motion, be it chaotic or otherwise, is confined to lie between invariant tori. As K is increased more KAM tori are destroyed. Eventually the only invariant tori that remain are those with noble winding numbers. The last torus to be destroyed is the one with winding number equal to the golden mean $\omega = \frac{\sqrt{5}-1}{2}$. Once all the KAM tori are destroyed global chaos sets in, restricted only by islands existing about the stable M cycles.

Estimates for Global Chaos.

Green obtained an estimate of the value of K for which the last KAM torus disappears in the Standard Map by considering the stability of the M cycles that approximate the golden mean torus.

He introduced the idea of a **Residue**

$$R_i = \frac{1}{4}\left(2 - trace\left(T_i^M\right)\right) \quad \text{where} \quad T = \begin{pmatrix} 1 - K\cos\theta & 1 \\ -K\cos\theta & 1 \end{pmatrix} \quad \text{trace } = 2 - K\cos\theta$$

Now the Residue R_i is related to the stability of the M_i cycle, because the its stability is determined from the $trace\left(T_i^M\right)$. In fact for a critical point

$$\begin{array}{lll} \text{if} \quad -2 < \text{trace}(T) < 2 & & \text{the critical point is stable (elliptic)} \\ \text{if} \quad \text{trace}(T) > 2 \ \text{ or } \ < -2 & & \text{the critical point is unstable (hyperbolic)} \end{array}$$

In terms of $R_i = \frac{1}{4}\left(2 - trace\left(T_i^M\right)\right)$

$$\text{if} \quad 0 < R < 1 \qquad \text{the critical point is stable (elliptic)}$$

$$\text{So the } M_i \text{ cycle is stable if} \quad 0 < R_i < 1$$

Now consider the limit of R_i over i, as the M_i cycles approach the irrational torus. Green showed that

$$\text{if} \ \lim_{i\to\infty} R_i = 0 \quad \text{the KAM curve still exists.}$$

$$\text{if} \ \lim_{i\to\infty} R_i \pm \infty \quad \text{the KAM curve has disappeared.}$$

Actually at the critical value $\lim_{i\to\infty} R_i$ is nonzero and finite.

Using this method Green found the critical value of K^* for the golden mean torus of the standard map

$$K^* = 0.9716354 \quad \text{for } K > K^* \text{ the map is glabally chaotic.}$$

For $K < K^*$ the golden mean torus bounds the motion.

However even once the torus has gone, strictly speaking, parts of it may remain for $K \approx K^*$ which provide a partial barrier to the flow. The torus is then called a cantorus. Reichl has more details on how to calculate the flow across these partial barriers.

Accelerator modes

Because the equation ($\theta_{n+1} = \theta_n + I_{n+1}$) for the angle θ is modulo 2π there are also solutions which are fixed in θ, but jump in action by integer multiples of 2π. Suppose $I_{n+1} = I_n - 2\pi\ell$, where ℓ is an integer. Then $\theta_{n+1} = \theta_n$ and since

$$I_{n+1} = I_n - K\sin\theta_n \qquad \Rightarrow \qquad K\sin\theta_n = 2\pi\ell$$

So that $\sin\theta_n = \dfrac{2\pi\ell}{K}$ which has a solution provided $K > 2\pi\ell$.

$$\sin\theta_{n\ell} = \frac{2\pi\ell}{K} \quad \forall n \quad I_{n\ell} = 2\pi(m - \ell n) \;\; \text{for} \;\; I_{1\ell} = 2\pi m$$

Stability

Since

$$T = \begin{pmatrix} 1 - K\cos\theta & 1 \\ -K\cos\theta & 1 \end{pmatrix} \qquad \text{is independent of } I$$

accelerator modes are stable if

$$-2 < 2 - K\cos\theta < 2 \qquad \Rightarrow \qquad 0 < K\cos\theta < 4$$

Since $K\sin\theta = 2\pi\ell > 0$ the mode with $\frac{\pi}{2} < \theta < \pi$ is always unstable. But the mode with $0 < \theta < \frac{\pi}{2}$ is stable if $K < \sqrt{16 + (2\pi\ell)^2}$. So if

$$2\pi\ell < K < 2\pi\ell\sqrt{1 + \left(\frac{2}{\pi\ell}\right)^2} \qquad \text{the accelerator mode exists and is stable.}$$

They are hard to find because though stable their island of stability is very small. However accelerator modes are important because they are responsible for Levi Flights which can have dramatic effects on the diffusion.