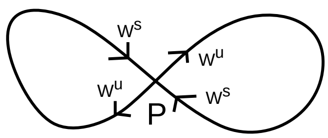
Section 1.9 Chaos near Homoclinic or Heteroclinic points.

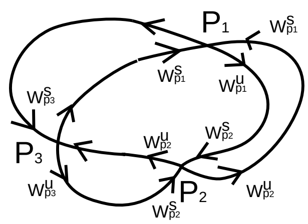
The separatrices which connect the saddles in say the nonlinear pendulum typically don't survive a perturbation. In fact the separatrix region is the first to go chaotic. Even in dissipative systems heteroclinic orbits (connections between different saddles) or homoclinic orbits (connections to the same saddle) are unlikely to survive a perturbation and are a common source of chaotic behavior.

Take an integrable system with a homoclinic orbit. The extensions of the eigenspaces of the saddle, the stable and unstable manifolds of the saddle, connect to each other. In a heteroclinic orbit the stable manifold of one saddle connects to the unstable manifold of the other saddle.

Two Homoclinic orbits

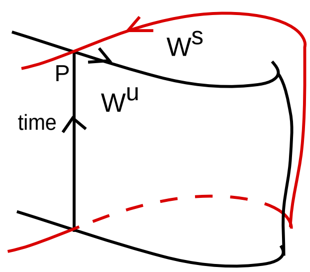


Heteroclinic orbits

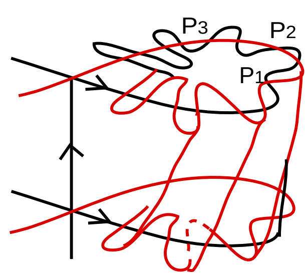


Now consider a canonical perturbation of such a system as a time periodic perturbation. Call the intersection points of the stable and unstable manifolds homoclinic or heteroclinic points depending on the unperturbed system. Suppose that under a perturbation T_ϵ the stable and unstable manifolds intersect at a point P_1. Then, because it takes an infinite amount of time to approach a saddle point, there will be an infinite number of intersection points.

Perturbed Homoclinic orbit
with time on the vertical axis.

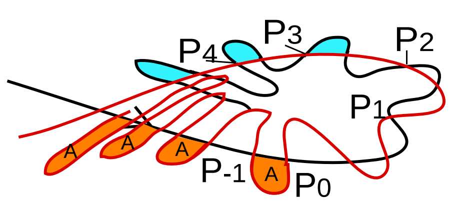


The intersecting manifolds



These homoclinic points are mapped forwards by T_ϵ and backwards by T_ϵ^{-1}. They become more and more closely spaced as the saddle point is approached. But since area is preserved by the map the oscillations in the manifold grow in amplitude. Near the saddle the intersections of the stable manifold and unstable manifold becomes very complicated.

The intersecting manifolds



Area preserving implies area A equal
but distance between succesive Pi tends to zero.

To investigate this we will first look at Melnikov's method for determining when and if transverse intersections of the manifolds do occur. Then we will use symbolic dynamics to investigate the actual dynamics of the map near the saddle point.

Melnikov's method for perturbations of Hamiltonian Systems.

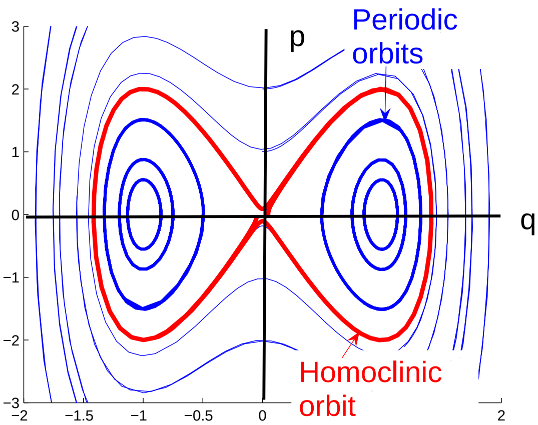
Suppose that

$$H(q,\, p,\, t) = H_0(q,\, p) + \epsilon H_1(q,\, p,\, t)$$

where H_0 has a homoclinic orbit surrounding a continuous family of periodic orbits $\mathbf{x}_\alpha$, with period T_α. The situation assumed is similar to the one for the Duffing Oscillator.

$$H_0(q,\, p) = \frac{p^2}{4} - 2q^2 + q^4 \quad \text{where} \quad \begin{array}{l} \dot{q} = \frac{p}{2} \\ \dot{p} = 4(q - q^3) \end{array}$$

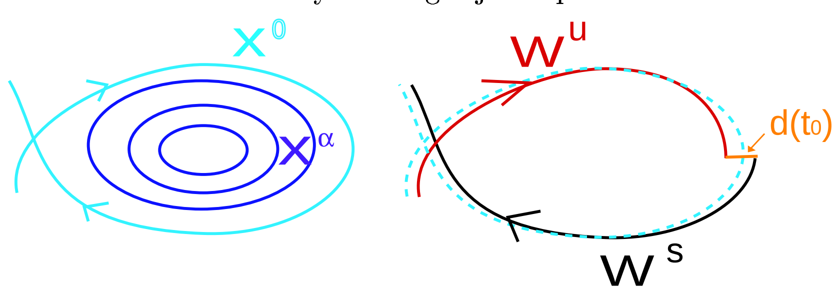
which has a saddle at the origin and two centers at $q = \pm 1$.



Further we require that the period of the orbits varies continuously with the energy. That is if $h_\alpha = H_0(\mathbf{x}_\alpha)$

$$\frac{dT_\alpha}{dh_\alpha} > 0 \quad \text{and} \quad T_\alpha \to 0 \quad \text{as the orbits approach the homoclinic orbit.}$$

Now perturb this system and there will still be saddle point near by, but the stable and unstable manifolds may no longer join up.



Suppose $\mathbf{x} = \begin{pmatrix} q \\ p \end{pmatrix} \in \mathbb{R}^2$ then in phase space

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(\mathbf{x},\, t) \quad \text{where} \quad \mathbf{f} = \begin{pmatrix} f_1(q,\, p) \\ f_2(q,\, p) \end{pmatrix} \quad \text{and} \quad \mathbf{g}(\mathbf{x},\, t + 2\pi) = \mathbf{g}(\mathbf{x},\, t)$$

If the unperturbed homoclinic in $\mathbb{R}^2$ is thought of as a curve parametrized by time, then at time t_0 the unstable and stable manifolds are separated by a perpendicular distance $d(t_0)$.

Let $\mathbf{x}^{(0)}(t-t_0)$ be the solution on the homoclinic orbit at time t.

The normal to the homoclinic orbit at time $t = t_0$ is

$$\mathbf{n}(\mathbf{x}^{(0)}(0)) = \left(\begin{array}{c} -f_2(\mathbf{x}^{(0)}(t-t_0)) \\ f_1(\mathbf{x}^{(0)}(t-t_0)) \end{array} \right)_{(t=t_0)}$$

Assuming that the perturbation is sufficiently smooth the perturbed manifolds will be close to the homoclinic orbit. So on the unstable manifold

$$\mathbf{x}^u_\epsilon(t,\, t_0) = \mathbf{x}^{(0)}(t-t_0) + \epsilon \mathbf{x}^u_1(t,\, t_0)$$

with a similar result for the stable manifold.

Then $d(t_0)$ is the component of $\mathbf{x}^u_\epsilon(t_0,\, t_0) - \mathbf{x}^s_\epsilon(t_0,\, t_0)$ in the direction of the normal $\mathbf{n}(\mathbf{x}^{(0)}(0))$.

If we define

$$\triangle(t,\, t_0) = \mathbf{n}\cdot(\mathbf{x}^u_1(t,\, t_0) - \mathbf{x}^s_1(t,\, t_0)) = \mathbf{f}(\mathbf{x}^{(0)}(t-t_0)) \wedge (\mathbf{x}^u_1(t,\, t_0) - \mathbf{x}^s_1(t,\, t_0))$$

where $\mathbf{a}\wedge\mathbf{b} = a_1 b_2 - a_2 b_1$, the actual distance in the direction of the normal is

$$d(t_0) = \frac{\epsilon \triangle (t_0,\, t_0)}{|\mathbf{f}(\mathbf{x}^{(0)}(0)|}$$

To work this out define $\triangle = \triangle^u - \triangle^s$ where

$$\triangle^u(t,\, t_0) = \mathbf{f}(\mathbf{x}^{(0)}(t-t_0)) \wedge \mathbf{x}^u_1(t,\, t_0) \quad \text{and} \quad \triangle^s(t,\, t_0) = \mathbf{f}(\mathbf{x}^{(0)}(t-t_0)) \wedge \mathbf{x}^s_1(t,\, t_0)$$

Differentiating gives

$$\frac{d\triangle^u}{dt}(t,\, t_0) = \left(D\mathbf{f}\dot{\mathbf{x}}^{(0)}\right) \wedge \mathbf{x}^u_1(t,\, t_0) + \mathbf{f}(\mathbf{x}^{(0)}(t-t_0)) \wedge \dot{\mathbf{x}}^u_1(t,\, t_0).$$

But $\dot{\mathbf{x}}^{(0)}(t-t_0) = \mathbf{f}(\mathbf{x}^{(0)}(t-t_0))$ and using Taylor series

$$\dot{\mathbf{x}}^{(0)} + \epsilon\dot{\mathbf{x}}^u_1 = \mathbf{f}(\mathbf{x}^{(0)}(t-t_0)) + \epsilon D\mathbf{f}(\mathbf{x}^{(0)}(t-t_0))\mathbf{x}^u_1(t,\, t_0) + \epsilon\mathbf{g}(\mathbf{x}^{(0)}(t-t_0),\, t) + 0(\epsilon^2)$$

so that

$$\dot{\mathbf{x}}^u_1 = D\mathbf{f}(\mathbf{x}^{(0)}(t-t_0))\mathbf{x}^u_1(t,\, t_0) + \mathbf{g}(\mathbf{x}^{(0)}(t-t_0),\, t) + 0(\epsilon).$$

Using these results

$$\frac{d\triangle^u}{dt}(t,\, t_0) = \left(D\mathbf{f}(\mathbf{x}^{(0)})\mathbf{f}(\mathbf{x}^{(0)})\right) \wedge \mathbf{x}^u_1(t,\, t_0) +$$

$$+\mathbf{f}(\mathbf{x}^{(0)}) \wedge \left(D\mathbf{f}(\mathbf{x}^{(0)})\mathbf{x}^u_1(t,\, t_0) + \mathbf{g}(\mathbf{x}^{(0)}(t-t_0),\, t)\right) + 0(\epsilon)$$

A little algebra shows that

$$\frac{d\triangle^u}{dt}(t,\, t_0) = \quad \text{trace } D\mathbf{f}(\mathbf{x}^{(0)})\mathbf{f}(\mathbf{x}^{(0)}) \wedge \mathbf{x}^u_1 + \mathbf{f}(\mathbf{x}^{(0)}) \wedge \mathbf{g}(\mathbf{x}^{(0)}(t-t_0),\, t) + 0(\epsilon)$$

But for Hamiltonian systems trace$D\mathbf{f} = 0$, so that

$$\frac{d\triangle^u}{dt}(t,\, t_0) = \mathbf{f}(\mathbf{x}^{(0)}) \wedge \mathbf{g}(\mathbf{x}^{(0)}(t - t_0),\, t) + 0(\epsilon)$$

and so finally

$$\triangle^u(t_0,\, t_0) - \triangle^u(t_0,\, -\infty) = \int_{-\infty}^{t_0} \mathbf{f}(\mathbf{x}^{(0)}) \wedge \mathbf{g}(\mathbf{x}^{(0)}(t - t_0),\, t)dt$$

Similarly

$$\triangle^s(\infty,\, t_0) - \triangle^s(t_0,\, t_0) = \int_{t_0}^{\infty} \mathbf{f}(\mathbf{x}^{(0)}) \wedge \mathbf{g}(\mathbf{x}^{(0)}(t - t_0),\, t)dt$$

Now

$$\triangle^s(\infty,\, t_0) = \lim_{t\to\infty} \triangle^s(t,\, t_0) = \lim_{t\to\infty} \mathbf{f}(\mathbf{x}^{(0)}(t - t_0)) \wedge \mathbf{x}_1^s(t,\, t_0)$$

which is zero because at the critical point $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

Similarly $\triangle^u(t_0,\, -\infty) = 0$.

Now the unnormalized distance is $\triangle(t_0,\, t_0) = \triangle^u(t_0,\, t_0) - \triangle^s(t_0,\, t_0)$ which is called the Melnikov function:

$$M(t_0) = \triangle(t_0,\, t_0) = \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{x}^{(0)}(t - t_0)) \wedge \mathbf{g}(\mathbf{x}^{(0)}(t - t_0),\, t)dt \qquad \textbf{Melnikov Function}$$

The actual perpendicular distance between the stable and unstable manifolds is

$$d(t_0) = \frac{\epsilon M(t_0)}{|\mathbf{f}(\mathbf{x}^{(0)}(0)|}$$

however if we are interested in the zeros of $d(t_0)$ then these are the same as the zeros of the Melnikov function, $M(t_0)$.

Example Forced Duffing.

$$H_0(q,\, p) = \frac{p^2}{4} - 2q^2 + q^4 - \epsilon q \cos \omega t \quad \text{where} \quad \begin{array}{l} \dot{q} = \frac{p}{2} \\ \dot{p} = 4(q - q^3) + \epsilon \cos \omega t \end{array}$$

If $\epsilon = 0$ the solution on the separatrix is

$$\begin{array}{l} q^{(0)} = \sqrt{2}\sec h(\sqrt{2}t) \\ p^{(0)} = 4\sec h(\sqrt{2}t)\tanh(\sqrt{2}t) \end{array} \quad \text{where at } t = 0 \quad \begin{array}{l} q^{(0)}(0) = \sqrt{2} \\ p^{(0)}(0) = 0 \end{array}$$

So the Melnikov Function is

$$M(t_0) = \int_{-\infty}^{\infty} \frac{p^{(0)}(t)}{2} \epsilon \cos \omega(t - t_0)dt.$$

This can be evaluated by the method of residues to give

$$M(t_0) = \pi\omega\epsilon \sec h \frac{\pi\omega}{2\sqrt{2}} \sin(\omega t_0)$$

This has an infinite set of simple zeros at the zeros of the sine function. Hence the stable and unstable manifolds intersect transversely at an infinite number of points.

If both the original and perturbed system are Hamiltonian then the Melnikov function can be written in terms of the Poisson Bracket:

$$M(t_0) = \int_{-\infty}^{\infty} \{H_0(q(t-t_0),\, p(t-t_0)), H_1(q(t-t_0),\, p(t-t_0),\, t)\}dt$$

or changing the time integration

$$M(t_0) = \int_{-\infty}^{\infty} \{H_0(q(t),\, p(t)), H_1(q(t),\, p(t),\, t+t_0)\}dt$$

Theorem

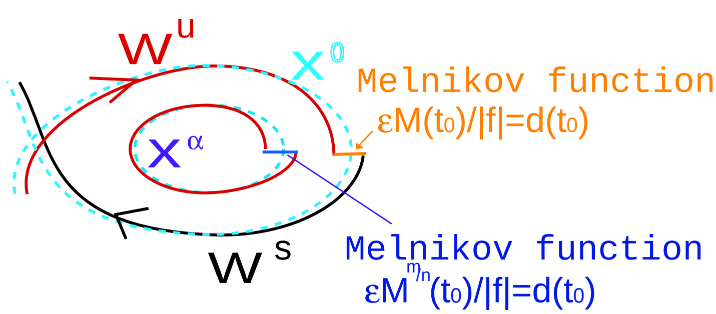
If $M(t_0)$ has simple zeros and is independent of ϵ, then for $\epsilon > 0$ sufficiently small, the unstable manifold of the perturbed saddle W_ϵ^u and the stable manifold of the perturbed saddle W_ϵ^s intersect transversely.

Melnikov's method can also be used to investigate perturbed orbits (not necessarily homoclinic to a critical point). Consider the periodic orbit with period $T_\alpha = \dfrac{mT}{n}$, where T is the period of the driver. Then

$$M^{\frac{m}{n}}(t_0) = \int_0^{mT} \mathbf{f}(\mathbf{x}^\alpha(t)) \wedge \mathbf{g}(\mathbf{x}^\alpha(t),\, t+t_0)dt$$

Theorem

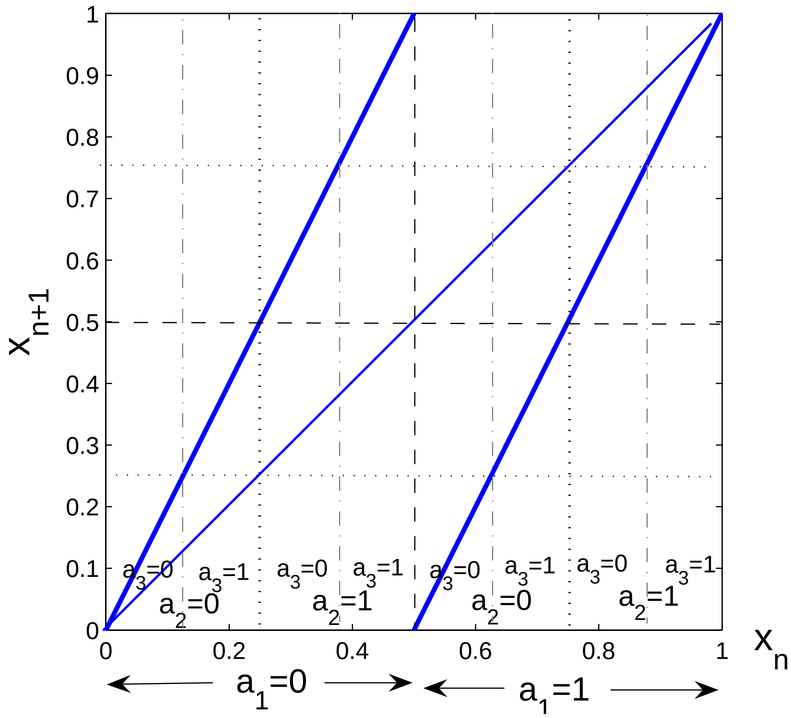
If $M^{\frac{m}{n}}(t_0)$ has simple zeros and is independent of ϵ and $\dfrac{dT_\alpha}{dh_\alpha} \neq 0$ then for $\epsilon > 0$ sufficiently small, there is a subharmonic orbit of period mT. So that the resonant closed curve of period T_α of the unperturbed Poincare Map breaks into a set of $2k = \frac{j}{m}$ periodic orbits each of period m, k of which are hyperbolic and k of which are elliptic.



The one dimensional Bakers Map. is very simple to write: $T(x) = 2x(mod 1)$.

$$T(x) = \begin{cases} 2x & \text{if } x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \le x < 1 \end{cases} \qquad \text{for } x \in [0,\ 1]$$

It maps onto $[0,\ 1]$.



Also the region $[0,\ \frac{1}{4}]$ is mapped to $[0,\ \frac{1}{2}]$ and the region $[\frac{1}{4},\ \frac{1}{2}]$ is mapped to $[\frac{1}{2},\ 1]$. So it seems likely that we could develop symbolic dynamics for this map as well. But because the map is comprised of straight lines the subintervals will all have equal length. In fact they have length $\dfrac{1}{2^n}$ and we can use the binary representation of points in $[0,\ 1]$ to develop the symbolic dynamics. Let $x \in [0,\ 1]$ then

$$x = \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \frac{a_4}{16} + \cdots = \sum_{i=1}^{\infty} \frac{a_i}{2^i} \qquad \text{for some } a_i = 0 \text{ or } 1$$

For instance if $x_0 = \frac{1}{3} < \frac{1}{2}$ then $a_1 = 0$, but since $\frac{1}{3} > \frac{1}{4}$ then $a_2 = 1$.
Now the remainder $\frac{1}{3} - \frac{1}{4} = \frac{1}{12} < \frac{1}{8}$, but $\frac{1}{12} > \frac{1}{16}$ so that $a_3 = 0$, $a_4 = 1$ etc.

$$\frac{1}{3} = \frac{0}{2} + \frac{1}{4} + \frac{0}{8} + \frac{1}{16} + \cdots$$

In this way any point in $[0,\ 1]$ can be represented in binary form.

Further the binary form relates directly to the map. The region $a_1 = 0$ has two subregions; $a_2 = 0$ in which points start in $a_1 = 0$ and are mapped to $a_1 = 0$ and $a_2 = 1$ in which points start in $a_1 = 0$ but are mapped to $a_1 = 1$.

So $[0,\ 1]$ could be divided into 8 the intervals of width $\frac{1}{8}$ labeled by $|a_0\, a_1\, a_2|$

$$|\, 000 \,|\, 001 \,|\, 010 \,|\, 011 \,|\, 100 \,|\, 101 \,|\, 110 \,|\, 111 \,|$$

The labels are simply the numbers 0, 1, 2, 3, 4, 5, 6, 7 in binary.

Iterating the Bakers map.
If

$$x_0 = \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \frac{a_4}{16} + \cdots + \frac{a_i}{2^i} \cdots \qquad \text{and} \qquad T(x_0) = \begin{cases} 2x_0 & \text{if } \ a_1 = 0 \\ 2x_0 - 1 & \text{if } \ a_1 = 1 \end{cases}$$

$$\text{then} \Longrightarrow \qquad T(x_0) = \begin{cases} 2\left(\frac{0}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \frac{a_4}{16} + \cdots\right) & \text{if } \ a_1 = 0 \\ 2\left(\frac{1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \frac{a_4}{16} + \cdots\right) - 1 & \text{if } \ a_1 = 1 \end{cases}$$

$$\Longrightarrow \qquad T(x_0) = \frac{a_2}{2} + \frac{a_3}{4} + \frac{a_4}{8} + \frac{a_5}{16} + \cdots + \frac{a_{i+1}}{2^i} + \cdots$$

So if

$$a(x_0) = [a_1\, a_2\, a_3\, a_4\, a_5 \ldots] \qquad \text{then} \qquad a(T(x_0)) = [a_2\, a_3\, a_4\, a_5 \ldots] \qquad \text{is a shift on symbols.}$$

The symbolic representation makes finding periodic orbits easy.
For instance the period-2 points have symbol sequences $[010101\ldots] = [\overline{01}]$ and $[\overline{10}]$.

But in the Bakers Map we can actually work out the x value that corresponds to that symbol sequence:

$$a(x) = [\overline{01}] \qquad \rightarrow \qquad x = \frac{0}{2} + \frac{1}{4} + \frac{0}{8} + \frac{1}{16} + \cdots = \sum_{j=1}^{\infty} \frac{1}{4^j} = \frac{1}{1 - \frac{1}{4}} - 1 = \frac{1}{3}$$

It is straightforward to check that this really is a period-2 point: $T(\frac{1}{3}) = \frac{2}{3}$ and $T(\frac{2}{3}) = \frac{1}{3}$.

Or take the period-3 point with symbol sequence $a(x) = [\overline{010}]$

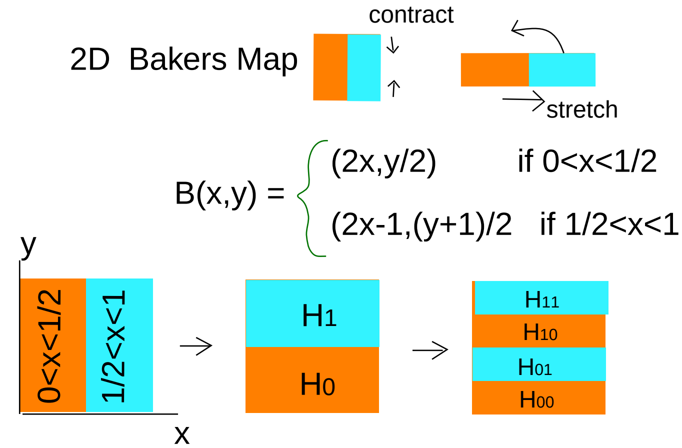
$$x = \frac{0}{2} + \frac{1}{4} + \frac{0}{8} + \frac{0}{16} + \frac{1}{32} + \cdots = \frac{1}{4} \sum_{j=0}^{\infty} \frac{1}{8^j} = \frac{1}{4} \frac{1}{1 - \frac{1}{8}} = \frac{2}{7}$$

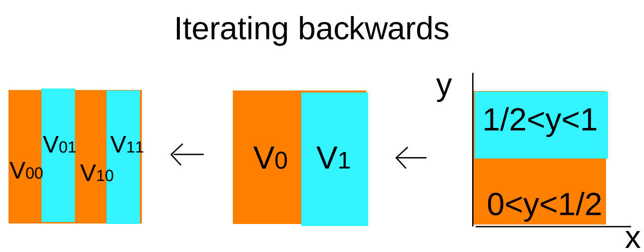
Check $T(\frac{2}{7}) = \frac{4}{7}, T(\frac{4}{7}) = \frac{1}{7}$ and $T(\frac{1}{7}) = \frac{2}{7}$.

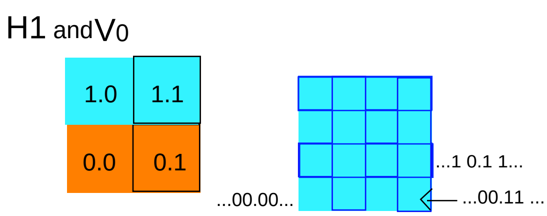
Properties of the Bakers Map

- 1) There are periodic orbits of all periods.
  In fact there are 2^n periodic points of period-n. For instance there are 8 period-3 points, two of which are critical points which leaves two period-3 orbits.
- 2) Sensitive dependence on initial conditions.
  This follows from the symbolic dynamics.
- 3) Periodic points are dense in [0, 1]. —— That is there is a periodic point arbitrarily close to any point in [0, 1].
- 4) If x and y are close then $|T(x) - T(y)|$ grows as $2|x - y|$ and $|T^n(x) - T^n(y)|$ grows as $2^n|x - y|$.
- 5) T has a dense orbit.—— This is a orbit which passes arbitrarily close to any point.

The 2D bakers Map is an area preserving Map in 2D whose iterates can be analyzed using symbolic dynamics. The name comes from kneading dough because the unit square is stretched in one direction and contracted in the other. The extended part is then cut and placed on top.







Each point in the unit square can be represented by a bi-infinite sequence.

$$a = \{...a_{-2}a_{-1}\cdot a_0 a_1 a_2 ...\} \quad \text{where the } a_i \text{ are 0 or 1.}$$

which represents the intersection of the vertical and horizontal strips in the limit.
The actual representation of points in $[0, 1] \times [0, 1]$ is very similar to that for the 1d Bakers Map. In fact we use a binary representation of points in $[0, 1]$ for both x and y

$$x = \sum_{i=1}^{\infty} \frac{a_{i-1}}{2^i} \qquad y = \sum_{i=1}^{\infty} \frac{a_{-i}}{2^i} \qquad \text{for some } a_i = 0 \text{ or } 1$$

Then

$$B(x,y) = \left\{ \begin{array}{ll} \left(\sum_{i=1}^{\infty} \frac{a_i}{2^i}, \sum_{i=2}^{\infty} \frac{a_{-(i-1)}}{2^i}\right) & \text{for } 0 < x < \frac{1}{2} \Rightarrow a_0 = 0 \\ \left(\sum_{i=1}^{\infty} \frac{a_i}{2^i} + a_0 - 1, \sum_{i=2}^{\infty} \frac{a_{-(i-1)}}{2^i} + \frac{1}{2}\right) & \text{for } 0 < x < \frac{1}{2} \Rightarrow a_0 = 1 \end{array} \right]$$

Now if $a_0 = 0$ effectively $a_{-1} = 0$ and if $a_0 = 1$ effectively $a_{-1} = 1$ and this term can be included in the sum for y. So

$$B(x,y) = \left(\sum_{i=1}^{\infty} \frac{a_i}{2^i}, \sum_{i=1}^{\infty} \frac{a_{-(i-1)}}{2^i} \right)$$

and to iterate the map we simply shift the decimal point. So that if

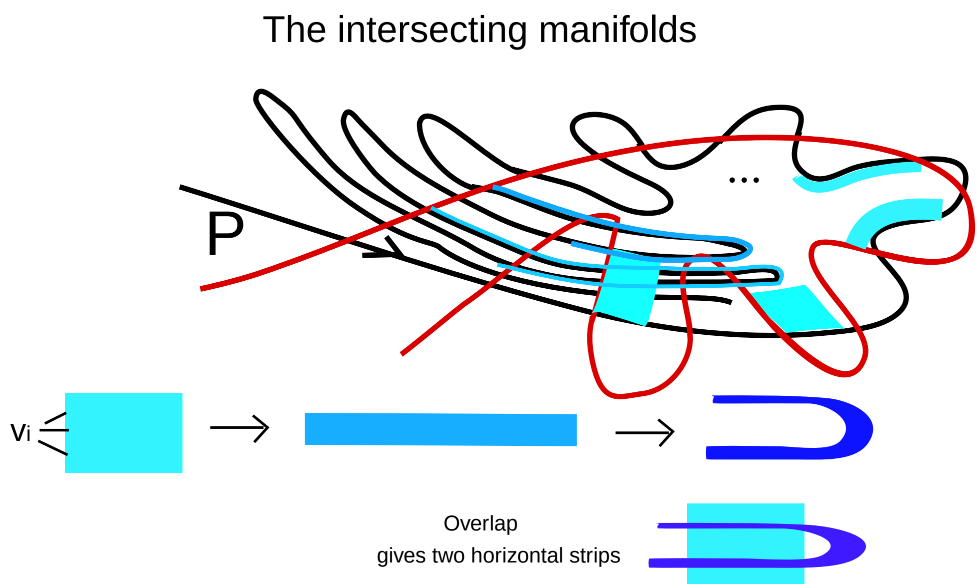
$$a(x,\, y) = \{...a_{-3}a_{-2}a_{-1}\cdot a_0 a_1 a_2 a_3...\} \quad \Rightarrow \quad a(B(x,\, y)) = \{...a_{-3}a_{-2}a_{-1}a_0 \cdot a_1 a_2...\}$$

The 1D and 2D Bakers Map has the following properties

- 1) There are periodic orbits of all periods and periodic points are dense.
- 2) There are an uncountable infinity of bounded non periodic orbits.
- 3) The map has a dense orbit.
- 4) The map has sensitivity to initial conditions.

The Horseshoe Map in 2D

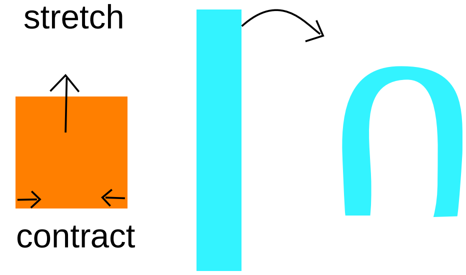
The Dynamics near the separatrix is actually quite similar to the 2D Bakers Map. If you take a square near the saddle point at the origin and iterate it forwards in time it will be stretched on one direction and contracted in the other. Eventually this elongated region returns to the neighborhood of the critical point and intersects with the original square. The result is a horseshoe map. The points that remain after all iterations can , like the 2D Bakers Map, be represented by a bi-infinite sequence.

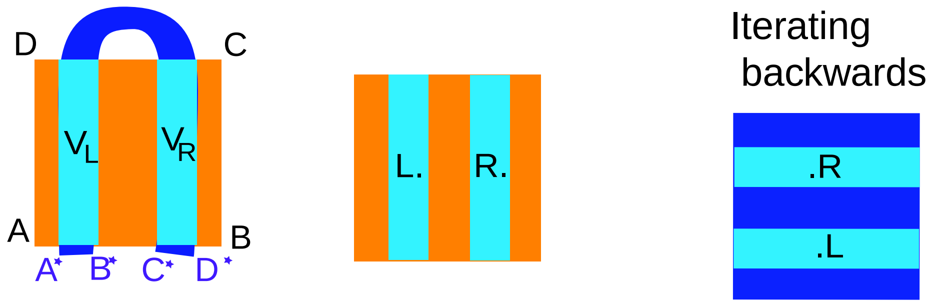


Unfortunately the folding that occurs in the horseshoe map makes the symbol sequences harder to write down. The situation is similar to the Logistic Map or Tent Map.

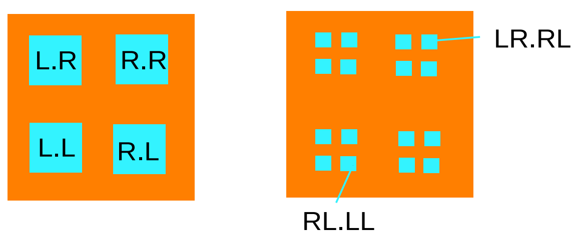
Consider the square ABCD, which we denote W, stretched and folded to become $A^*B^*C^*D^*$, denoted $h(W)$, which is shaped like a horseshoe. Some parts of ABCD are mapped out of ABCD. We will only consider those points that remain, which are $W \cap h(W)$ and comprise of two vertical strips V_L and V_R. On the second iteration more points are mapped out, those that remain $W \cap h^2(W)$. The points that remain for all iterations of the map, forwards and backwards forms a cantor set Λ.

Horseshoe Map





So that the points are denoted by a bi-infinte sequence of R's and L's.



Now, as with the Bakers map for any point v in Λ we assign a bi-infinite symbol sequence

$$a(v) = \{...a_{-3}a_{-2}a_{-1}.a_0a_1a_2a_3...\} \quad \text{where if} \begin{array}{lll} h^i(v) & \text{lies in } V_L \text{ let} & a_i = L. \\ h^i(v) & \text{lies in } V_R \text{ let} & a_i = R. \end{array}$$

However this only uses a_i for $i \geq 1$ and so only fixes the point horizontally.

To fix the point vertically we need to iterate backwards. This gives horizontal strips which we denote $.L$ and $.R$. From this we obtain the regions $L.L$, $L.R$, $R.L$ and $R.R$. The ordering of the strips is not as straight forward as for the Bakers map.

The horseshoe map has an invariant Cantor set Λ such that

- 1) There are periodic orbits of all periods and periodic points are dense in Λ.
- 2) There are an uncountable infinity of bounded non periodic orbits in Λ.
- 3) The map has a dense orbit in Λ.
- 4) The map has sensitivity to initial conditions.

Smale-Birchoff Homoclinic Theorem.

Consider a map $f : \mathbb{R}^2 -> \mathbb{R}^2$ with an unstable (saddle-like) fixed point at P. The stable and unstable manifolds of P are defined as

$$W_P^s = \{x : f^n(x) \to P \quad \text{as} \quad n \to \infty\}$$

$$W_P^u = \{x : f^n(x) \to P \quad \text{as} \quad n \to -\infty\}.$$

A **homoclinic tangle** occurs when these two manifolds intersect transversely. The set of points $W_P^s \cap W_P^u$, called **homoclinic points**, asymptotically attract to P as $n \to \pm\infty$.

> **Theorem.** Let $f : \mathbb{R}^2 -> \mathbb{R}^2$ be a diffeomorphism (a continuously differentiable function that has a continuously differentiable inverse). Let y be a transversal homoclinic point. Then there is an integer n such that $F = f^n$ has a hyperbolic invariant set Λ homeomorphic to a Cantor set containing P and y. The periodic and homoclinic points are dense in Λ and $F|_\Lambda$ is topologically conjugate to a shift acting on the space of bi-infinite symbol sequences.

This theorem can then be used in conjunction with Melnikov's method for proving the existence of transversal homoclinic points.