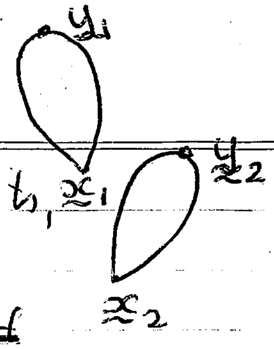
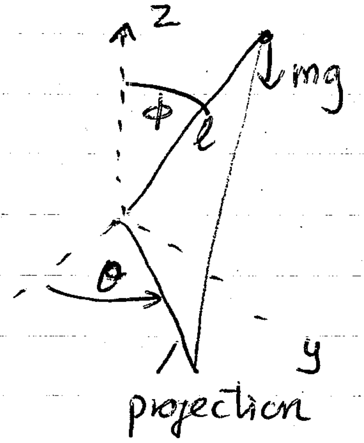
Math 4104 Solns to tute 1



1a) Each juggling pin is described by two points, x_i, y_i $i = 1, 2$ $\Rightarrow$ 2×6 degrees of freedom $= \underline{\underline{12 \text{ dof}}}$

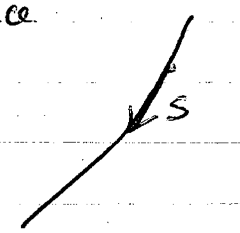
Configuration space (x_1, y_1, x_2, y_2)

b) <u>Spherical Pendulum</u>



The bob moves on a sphere radius l. So the position of the point mass can be described by two angles: ϕ the angle the rod makes with the vertical and θ the angle the projection of the rod onto the (x, y) plane makes with the x axis.

$-$ $\underline{\underline{2 \text{ dof}}}$ Configuration space (θ, ϕ)



c) The wire is parametrised by s then this is the configuration space

$-$ $\underline{\underline{1}}$ dof

2a) $\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \left\{ \frac{x^2+y^2}{2} + x^2 y + \frac{y^3}{3} \right\}$

Lagranges eqns of motion $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$ where $q_1 = x$, $q_2 = y$

For x: $\frac{d}{dt}(m\dot{x}) + x + 2xy = 0 \Rightarrow m\ddot{x} = -x - 2xy$

y: $\frac{d}{dt}(m\dot{y}) + y + x^2 + y^2 = 0 \Rightarrow m\ddot{y} = -y - x^2 - y^2$

2b) $\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \left\{\frac{x^2+y^2}{2} - \frac{x^4}{4}\right\}$

Lagranges equations of motion $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$

So for x: $\quad \frac{d}{dt}(m\dot{x}) + x - x^3 = 0 \quad \Rightarrow \quad m\ddot{x} = -x + x^3$

y: $\quad \frac{d}{dt}(m\dot{y}) + y = 0 \quad \Rightarrow \quad m\ddot{y} = -y$ //

c) Using diagram in 1b) with $l = R$

$x = R\sin\theta\cos\varphi \;, \; y = R\sin\theta\sin\varphi \;, \; z = R\cos\varphi$

$\mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$\mathcal{L}(\theta, \varphi, \dot{\theta}, \dot{\varphi}) = \frac{m}{2}R^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta)$

Euler-Lagrange eqns of motion are

θ: $\quad \frac{d}{dt}\left(\frac{mR^2 2\dot{\theta}}{2}\right) = mR^2\ddot{\theta} = \frac{m}{2}R^2\dot{\varphi}^2 2\sin\theta\cos\theta$

$\Rightarrow \quad \ddot{\theta} = \frac{1}{2}\dot{\varphi}^2\sin 2\theta$

φ: $\quad \frac{d}{dt}\left(mR^2\dot{\varphi}\sin^2\theta\right) = 0 \quad \Rightarrow \quad \dot{\varphi}\sin^2\theta = \text{constant}$ //

d) One dof $\Rightarrow$ 1 Euler-Lagrange eqn:

$\frac{d}{dt}\left(e^{\alpha t}\dot{q}\right) = -\omega^2 q e^{\alpha t}$

$\Rightarrow \quad e^{\alpha t}(\alpha\dot{q} + \ddot{q}) = -e^{\alpha t}\omega^2 q$

or $\quad \ddot{q} = -\alpha\dot{q} - \omega^2 q \quad$ (Damped linear oscillator)

3 From 2b) the Kinetic energy for the spherical pendulum is $\frac{1}{2}mR^2(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta)$

The potential energy is given by $mgR\cos\theta$

So $L(\theta,\varphi,\dot{\theta},\dot{\varphi}) = \frac{1}{2}mR^2(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta) - mgR\cos\theta$

Since L is not a function of φ $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) = 0$

$\Rightarrow \frac{\partial L}{\partial\dot{\varphi}} = mR^2\dot{\varphi}\sin^2\theta = \text{constant}$, is not a function of φ,

Similarily the Euler Lagrange equation for θ is not a function of φ. So the system is symmetric under rotations in φ. This means that $\varphi' = \varphi + \alpha$ leaves L unchanged. So $\frac{\partial L}{\partial\alpha} = 0 \Rightarrow \frac{\partial L}{\partial\dot{\theta}}\frac{\partial\theta}{\partial\alpha} + \frac{\partial L}{\partial\dot{\varphi}}\frac{\partial\varphi}{\partial\alpha}$ is a conserved quantity.

Now $\frac{\partial L}{\partial\dot{\theta}}\frac{\partial\theta}{\partial\alpha} + \frac{\partial L}{\partial\dot{\varphi}}\frac{\partial\varphi}{\partial\alpha} = \frac{\partial L}{\partial\dot{\varphi}} = mR^2\sin^2\theta\,\dot{\varphi}$

$\Rightarrow \dot{\varphi}\sin^2\theta$ is a conserved quantity.

4 a) If $L(t:q,\dot{q}) = \frac{1}{2}e^{\alpha t}(\dot{q}^2 - \omega^2q^2)$

then the conjugate momentum $p = \frac{\partial L}{\partial\dot{q}} = e^{\alpha t}\dot{q}$

So the Hamiltonian $H(t,q,p) = p\dot{q}(p) - \frac{1}{2}e^{\alpha t}((\dot{q}(p))^2 - \omega^2q^2)$

Now $\dot{q} = pe^{-\alpha t}$ $\quad = \frac{1}{2}e^{-\alpha t}p^2 + \frac{1}{2}e^{\alpha t}\omega^2q^2$

Hamiltons equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p} = e^{-\alpha t}p$$

$$\dot{p} = \frac{-\partial H}{\partial q} = -e^{\alpha t}\omega^2 q$$

4b) $p_\theta = \frac{\partial L}{\partial \dot\theta} = \frac{\partial L}{\partial \alpha} = mR^2\alpha$, $p_\varphi = \frac{\partial L}{\partial \dot\varphi} = \frac{\partial L}{\partial \beta} = mR^2\beta \sin^2\theta$

$\Rightarrow \quad \alpha = \frac{p_\theta}{mR^2}$, $\beta = \frac{p_\varphi}{mR^2\sin^2\theta}$

$$H(t, \theta, \varphi, p_\theta, p_\varphi) = \frac{\left(p_\theta^2 + p_\varphi^2/\sin^2\theta\right)}{2mR^2}$$

Hamiton's equations of motion

$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2}$ $\qquad \dot{p_\theta} = -\frac{\partial H}{\partial \theta} = \frac{-p_\varphi^2\cos\theta}{mR^2\sin^3\theta}$

$\dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mR^2\sin^2\theta}$ $\qquad \dot{p_\varphi} = \frac{-\partial H}{\partial \varphi} = 0$

4c) $L(x, \dot x) = m_0c^2\left(1 - \sqrt{1 - \frac{\dot x^2}{c^2}}\right) - \frac{1}{2}m_0\Omega^2x^2$

Conjugate momentum

$$p = \frac{\partial L}{\partial \dot x} = \frac{m_0\dot x}{\sqrt{1 - \dot x^2/c^2}}$$

So solving for $\dot x$;

$$m_0^2\dot x^2 = p^2(1 - \dot x^2/c^2)$$

$\Rightarrow \quad \dot x^2 = p^2/(m_0^2 + p^2/c^2)$ and $1 - \frac{\dot x^2}{c^2} = \frac{m_0^2}{m_0^2 + p^2/c^2}$

Hamiltonian

So $H = \dot x p - L(x, \dot x)$

$$= \frac{p^2}{\sqrt{m_0^2 + p^2/c^2}} - m_0c^2\left(1 - \frac{m_0}{\sqrt{m_0^2 + p^2/c^2}}\right) + \frac{1}{2}m_0\Omega^2x^2$$

$$= c\left(\sqrt{c^2m_0^2 + p^2} - cm_0\right) + \frac{1}{2}m_0\Omega^2x^2$$

Hamilton's eqns of motion

$\dot x = \frac{\partial H}{\partial p} = \frac{cp}{\sqrt{c^2m_0^2 + p^2}}$ $\qquad \dot p = -\frac{\partial H}{\partial x} = -m_0\Omega^2x$

5. If $L(q,\dot q) = \frac{\dot q^2}{2} + q\left(1-\frac{q^2}{3}\right)$

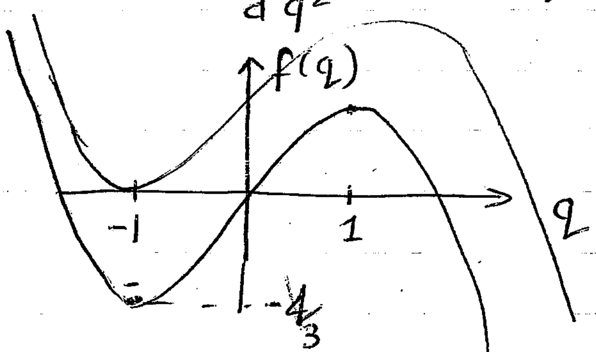
then $p = \frac{\partial L}{\partial \dot q} = \dot q \Rightarrow H = \frac{p^2}{2} - q\left(1-\frac{q^2}{3}\right)$

So $H = c \Rightarrow p^2 = 2q\left(1-\frac{q^2}{3}\right) + 2c$

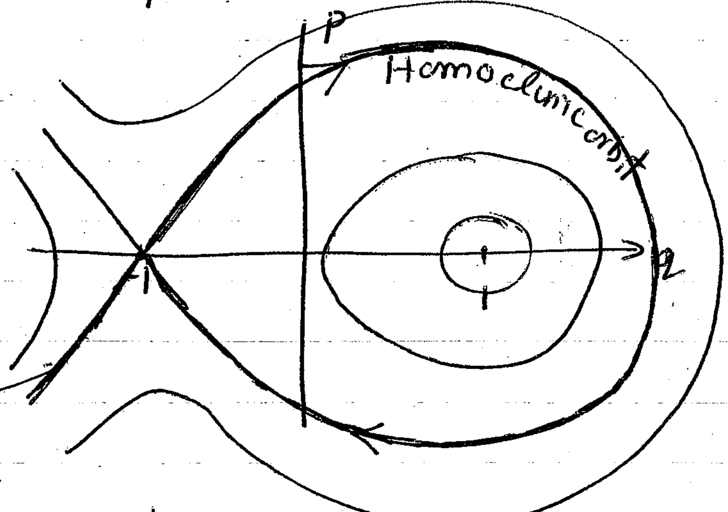
Consider the cubic $f(q) = 2q\left(1-\frac{q^2}{3}\right)$

$$\frac{df}{dq} = 2\left(1-q^2\right) = 0 \quad \text{at } q = \pm 1$$

$$\frac{d^2f}{dq^2} = -4q \quad \begin{matrix} >0 \text{ for } q=-1 & \text{local min} \\ <0 \quad\;\; q=+1 & \text{max} \end{matrix} \quad \begin{matrix} f(-1) = -\frac{4}{3} \\ f(1) = \frac{4}{3} \end{matrix}$$



Now $f(q) + \frac{4}{3}$ just touches the q axis at $q = -1$.



Taking the square root gives the integral curves. There is a saddle at $(-1,0)$ and a center at $(1,0)$

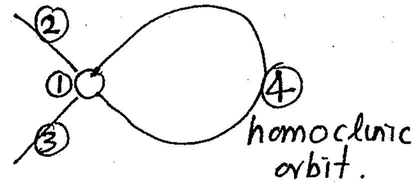
The homoclinic orbit

$$2H = \frac{4}{3} = -2q\left(1-\frac{q^2}{3}\right) + p^2$$

has the energy (H value) at the saddle.

Actually $H = \frac{2}{3} = \frac{p^2}{2} - q\left(1-\frac{q^2}{3}\right)$

is comprised of 4 trajectories:



The saddle $(-1,0)$, two for $q < -1$

and the homoclinic orbit.

The homoclinic orbit is also the unstable and the stable manifold of the saddle $(-1,0)$ for $q > -1$.

Because of this it has the property that as $t \to \infty$ $x \to (-1,0)$ and as $t \to -\infty$ $x \to (-1,0)$

6. a) Hamiltons equations of motion

$$\dot r = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu} \quad , \quad \dot p_r = \frac{-\partial H}{\partial r} = \frac{-k}{r^2} + \frac{p_\theta^2}{\mu r^3}$$

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2} \quad , \quad \dot p_\theta = \frac{-\partial H}{\partial \theta} = 0$$

b) Since H does not depend on θ the system is symmetric under rotations in θ ($\theta' = \theta + \alpha$ leaves H unchanged)

c) There is a conserved quantity arising from the symmetry which in terms of momenta is p_θ, since $\dot p_\theta = 0$

d) Let $p_\theta = \ell$ then $\dot p_r = -k/r^2 + \ell^2/\mu r^3$ is a function of r.

and $\dot\theta = \ell/\mu r^2$ is also "

So $\frac{dp_r}{d\theta} = \left(-\frac{k}{r^2} + \frac{\ell^2}{\mu r^3}\right)\left(\frac{\mu r^2}{\ell}\right) = -\frac{k\mu}{\ell} + \ell/r$ is "

And $\frac{dr}{d\theta} = \frac{\dot r}{\dot\theta} = \frac{p_r}{\mu}\left(\frac{\mu r^2}{\ell}\right) = \frac{p_r}{\ell} r^2$

$$\Rightarrow \frac{d^2 r}{d\theta^2} = \left(\frac{dp_r}{d\theta}\right)\frac{r^2}{\ell} + 2r\frac{p_r}{\ell}\frac{dr}{d\theta}$$

$$= \left(-\frac{k\mu}{\ell} + \ell/r\right)\left(r^2/\ell\right) + \frac{2r}{r^2}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{d^2 r}{d\theta^2} - 2/r\left(\frac{dr}{d\theta}\right)^2 + \frac{k\mu}{\ell^2} r^2 - r = 0$$

Now let $u = 1/r \Rightarrow \frac{du}{d\theta} = -1/r^2 \frac{dr}{d\theta}$, $\frac{d^2u}{d\theta^2} = \frac{-1}{r^2}\frac{d^2r}{d\theta^2} + \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2$

$$\frac{d^2u}{d\theta^2} = \frac{k\mu}{\ell^2} - u \quad \text{or} \quad \frac{d^2u}{d\theta^2} + u = \frac{k\mu}{\ell^2}$$

$$\Rightarrow u = A\cos(\theta - \theta_0) + k\mu/\ell^2 \Rightarrow r(\theta) = \frac{\ell^2}{k\mu\,(1 + \bar A\cos(\theta - \theta_0))}$$

where $H = \frac{k^2\mu}{2\ell^2}\bar A^2$

hyperbola	$\bar A > 1$
ellipse	$0 < \bar A < 1$
circle	$0 = \bar A$
parabola	$\bar A = 1$

$\Rightarrow$ hyperbola $H > \frac{k^2\mu}{2\ell^2}$

ellipse $0 < H < \frac{k^2\mu}{2\ell^2}$

circle $H = 0$ & parabola $H = \frac{k^2\mu}{2\ell^2}$

7. a) If $F(x)$ is related to $G(y)$ by a Legendre transform then $F(x) + G(y) = xy$

and $y = \frac{\partial F}{\partial x}$

Here $F(x) = ax + bx^2$ so $y = a + 2bx \Rightarrow x = \frac{y-a}{2b}$

$\Rightarrow xy = \frac{y^2 - ay}{2b}$

So $G(y) = \frac{y^2 - ay}{2b} - a\left(\frac{y-a}{2b}\right) - b\left(\frac{y-a}{2b}\right)^2$

$= \frac{y^2}{4b} - \frac{ay}{2b} + \frac{a^2}{4b}$

(Check $\frac{\partial G}{\partial y} = \frac{y}{2b} - \frac{a}{2b} = x$!)

b) $F(x,y)$ is related to $G(w,y)$ with y passive

$\Rightarrow F(x,y) + G(w,y) = xw$

and $w = \frac{\partial F}{\partial x}$

Here $w = +a \cancel{\cos} x \cos y \Rightarrow x = \arccos\left(\frac{w}{a\cos y}\right)$

So $G(w,y) = w \arccos\left(\frac{w}{a\cos y}\right) - a\cos y \sin\left(\arccos \frac{w}{a\cos y}\right)$

$= w \arccos\left(\frac{w}{a\cos y}\right) - a\cos y\sqrt{1 - \frac{w^2}{a^2\cos^2 y}}$

Check $\frac{\partial G}{\partial w} = \arccos \frac{w}{a\cos y} + \cancel{\frac{w}{a\cos y\sqrt{1 - \frac{w^2}{a^2\cos^2 y}}}}$

$+ \cancel{\frac{a\cos y}{\sqrt{1 \ldots}}} \times \left(\frac{-w}{a^2\cos^2 y}\right) = x$ ✓

$\frac{\partial G}{\partial y} = \cancel{\frac{-w^2 \sin y}{a\cos^2 y\sqrt{1 - \left(\frac{w^2}{a^2\cos^2 y}\right)}}} + a\sin y\sqrt{1 - \frac{w^2}{a^2\cos^2 y}} + \cancel{\frac{a\cos y\, w^2 \sin y}{a^2\cos^3 y\sqrt{\ }}}$

$\frac{\partial F}{\partial y} = -a\sin y \sin x = -\frac{\partial G}{\partial y}$

7c) $F(u,x) = \sqrt{1-u^2} - x^2$

$F(u,x)$ is related to $G(v,x)$ by a legendre transform with x passive $\Rightarrow$ $v = \frac{\partial F}{\partial u} = \frac{-u}{\sqrt{1-u^2}}$

Then $G(v,x) = v\,u(v) - F(u(v), x)$

where since $v = \frac{-u}{\sqrt{1-u^2}} \Rightarrow u = \frac{-v}{\sqrt{1+v^2}}$

$$G(v,x) = \frac{-v^2}{\sqrt{1+v^2}} - \sqrt{1-\frac{v^2}{1+v^2}} + x^2$$

$$= -\sqrt{1+v^2} + x^2$$

Check $\frac{\partial G}{\partial v} = \frac{-v}{\sqrt{1+v^2}} = u$ ✓

and $\frac{\partial G}{\partial x} = 2x = -\frac{\partial F}{\partial x}$ ✓

8. If I_1 and I_2 are two constants of the motion then

$$\{I_1, H\} = 0 \text{ and } \{I_2, H\} = 0$$

Now $\{\{a,b\},c\} = \{a,\{b,c\}\} - \{b,\{a,c\}\}$

So $\{\{I_1,I_2\},H\} = \{I_1,\{I_2,H\}\} - \{I_2,\{I_1,H\}\}$

$$= \{I_1, 0\} - \{I_2, 0\}$$

$$= 0 - 0 = 0 \;//$$

9. Euler Lagrange equations of motion

a) $\frac{d}{dt}\left(G(q,t)\,\dot q + F(q,t)\right) - \frac{\partial}{\partial q}\left(\frac{1}{2} G \dot q^2 + F\dot q - V\right) = 0$

$$\Rightarrow \frac{d}{dt}\left(G(q,t)\,\dot q\right) + \cancel{\frac{\partial F}{\partial q}\dot q} + \frac{\partial F}{\partial t} - \frac{1}{2}\frac{\partial G}{\partial q}\dot q^2 \cancel{-\frac{\partial F}{\partial q}\dot q} + \frac{\partial V}{\partial q} = 0$$

b) $\frac{d}{dt}\left(G(q,t)\,\dot q\right) + \frac{\partial^2 f}{\partial q \partial t} + \frac{\partial V}{\partial q} - \frac{1}{2}\frac{\partial G}{\partial q}\dot q^2 = 0$

But $\frac{\partial^2 f}{\partial q \partial t} = \frac{\partial F}{\partial t}$ so there equations are the same.

However the Hamiltonians are different:

a) $p = \frac{\partial L_a}{\partial \dot q} = G\dot q + F \Rightarrow \dot q(p) = (p-F)/G$

$$H_a = p\dot q(p) - \left(\frac{1}{2}G(\dot q(p))^2 + F\dot q(p) - V\right)$$

$$= p\frac{(p-F)}{G} - \frac{(p-F)^2}{2G} - F\frac{(p-F)}{G} + V$$

$$= \frac{(p-F)^2}{2G} + V$$

b) $p = G\dot q \Rightarrow \dot q = p/G$

$$H_b = p\dot q(p) - \frac{1}{2}G(\dot q(p))^2 + V + \frac{\partial f}{\partial t}$$

$$= \frac{p^2}{G} - \frac{p^2}{2G} + V + \frac{\partial f}{\partial t}$$

$$= \frac{p^2}{2G} + V(q,t) + \frac{\partial f}{\partial t}$$

Now in the example a) $F(q,t) = \cos(q+t)$, $V(q,t) = \sin(q+t)$

and in b) $\frac{\partial f}{\partial t} = \cos(q+t) \Rightarrow f = +\sin(q+t)$

$$\Rightarrow \frac{\partial f}{\partial q} = \cos(q+t) = F \text{ of a)}$$

So from the first part of the question there Hamiltonians give rise to the same motion.