MATH 4104/09 Tute 2

Q1: A Legendre transform for $F_2(P,q,t)$ to $F_4(P,p,t)$ will have (P,t) as passive variables and p and q as active variables.

Recall that for an F_2 generating function

$$p = \frac{\partial F_2}{\partial q} \quad \text{and so}$$

for a Legendre transform

$\exists$ a function G s.t. $q = \frac{\partial G}{\partial p}$

where $G(P,p,t) = pq - F_2(P,q,t)$

So let $F_4(P,p,t) = pq - F_2(P,q,t)$

then $\underline{\frac{\partial F_4}{\partial p} = q}$ and $\frac{\partial F_4}{\partial P} = \underline{-\frac{\partial F_2}{\partial P} = -Q}$

Now if $F_2(P,q) = q^2 e^P \Rightarrow F_4(P,p) = pq - q^2 e^P$

$\Rightarrow p = 2q e^P \qquad\qquad = \frac{p^2}{2} e^{-P} - \frac{p^2}{4} e^{-2P} e^P$

$\Rightarrow q = \frac{p}{2} e^{-P}$

$$F_4(P,p) = \frac{p^2}{4} e^{-P}$$

Q2. let $P = 1/q$ and $Q = pq^2$ then

$$H = \tfrac{1}{2}(P^2 + Q^2).$$

This is canonical because it can be generated by

$F_1(q,Q,t) = -Q/q \qquad P = -\frac{\partial F_1}{\partial Q}$

$p = \frac{\partial F_1}{\partial q} = \frac{Q}{q^2} \Rightarrow Q = pq^2\,!$

OR simply use Poisson Brackets

$$\{Q,P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 2pq \times 0 - q^2\left(\frac{-1}{q^2}\right) = 1$$

3 a) $\{\phi, J\} = \frac{\partial \phi}{\partial \theta}\frac{\partial J}{\partial I} - \frac{\partial \phi}{\partial I}\frac{\partial J}{\partial \theta}$

But note that $\phi = \theta + J = \theta + I + K\sin\theta$, $J = I + K\sin\theta$

$\{\phi, J\} = (1 + K\cos\theta)1 - 1(K\cos\theta) = 1$ ✓

Now the generating function F_2 satisfies

$$I = \frac{\partial F_2}{\partial \theta} \quad \text{and} \quad \phi = \frac{\partial F_2}{\partial J}$$

$$\Rightarrow J - K\sin\theta = \frac{\partial F_2}{\partial \theta} \quad \text{and} \quad \theta + J = \frac{\partial F_2}{\partial J}$$

$$\Rightarrow F_2 = J\theta + \frac{J^2}{2} + K\cos\theta$$

b) $\{Q_1, Q_2\} = \frac{\partial Q_1}{\partial q_1}\frac{\partial Q_2}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial p_2} - \frac{\partial Q_1}{\partial p_1}\frac{\partial Q_2}{\partial q_1} - \frac{\partial Q_1}{\partial p_2}\frac{\partial Q_2}{\partial q_2}$

$= 1\times 0 + 0\times 1 - 0\times 0 - 0\times 0 = 0$

Similarily $\{P_1, P_2\} = 0$, $\{Q_1, P_1\} = 1$, $\{Q_2, P_1\} = 0$
$\{Q_1, P_2\} = 0$ and $\{Q_2, P_2\} = 1$

The generating function is a combination of F_1 and F_2

$$F(q_1, P_1, q_2, Q_2) = (2q_1 + q_2)Q_2 + q_1 P_1$$

Then $p_1 = \frac{\partial F}{\partial q_1}$, $Q_1 = \frac{\partial F}{\partial P_1}$, $p_2 = \frac{\partial F}{\partial q_2}$, $P_2 = -\frac{\partial F}{\partial Q_2}$

gives $p_1 = 2Q_2 + P_1$, $Q_1 = q_1$, $p_2 = Q_2$, $P_2 = -(2q_1 + q_2)$

$\Rightarrow Q_1 = q_1$, $P_1 = p_1 - 2Q_2 = p_1 - 2p_2$

$Q_2 = p_2$, $P_2 = -2q_1 - q_2$ as required

5. $\{x, p\} = \frac{\partial x}{\partial \theta}\frac{\partial p}{\partial I} - \frac{\partial x}{\partial I}\frac{\partial p}{\partial \theta}$

$= \beta\gamma I^{\alpha}\cos\theta \, \frac{\beta\alpha}{\gamma} I^{\alpha-1}\cos\theta - \beta\gamma\alpha I^{\alpha-1}\sin\theta\left(-\frac{\beta}{\gamma}I^{\alpha}\sin\theta\right)$

$= \beta^2\alpha I^{2\alpha-1}(\cos^2\theta + \sin^2\theta) = \beta^2\alpha I^{2\alpha-1} = 1$

So $2\alpha = 1 \Rightarrow \underline{\alpha = \tfrac{1}{2}}$ and $\beta^2\alpha = 1 \Rightarrow \underline{\beta = \pm\sqrt{2}}$

$\underline{\gamma \text{ can be any number}}$.

6. Using a near identity generating function.

$$F_2(\theta_i, J_i) = J_1\theta_1 + J_2\theta_2 + \epsilon g(\theta_i, J_i)$$

$\Rightarrow$ the old actions are given by

$$I_i = \frac{\partial F_2}{\partial \theta_i} = J_i + \epsilon\frac{\partial g}{\partial \theta_i}$$

and the new angles are given by

$$\phi_i = \frac{\partial F_2}{\partial J_i} = \theta_i + \epsilon\frac{\partial g}{\partial J_i}$$

So that the Hamiltonian in new variables is

$$H(\phi_i, J_i) = J_1 + \epsilon\frac{\partial g}{\partial \theta_1} + J_2 + \epsilon\frac{\partial g}{\partial \theta_2} + \epsilon(J_2\cos\theta_1 + J_1^2)$$

up to order ϵ.

Now we require that $H(J_i)$ only up to order ϵ

There is no need to remove the ϵJ_1^2 term as it is a function of the action only. Suppose we remove $\epsilon J_2\cos\theta_1$ term then $H = J_1 + J_2 + \epsilon J_1^2$ is an integrable system.

Also the ϵJ_1^2 cannot be removed if the solution is to be valid for all θ as $g \sim \theta\epsilon J_1^2$ and $\theta\epsilon$ is not small for all θ.

3c) Use matrices

$$Q_1 = q_1 - 5q_2 \qquad Q_2 = q_2$$
$$P_1 = p_1 \qquad P_2 = p_2 + 5p_1$$

$$A = \begin{pmatrix} 1 & -5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A J_4 A^T = A \begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 5 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = J_4 \text{ as required} \Rightarrow \text{canonical}$$

Since $p_1 = \frac{\partial F_2}{\partial q_1}$ & $p_2 = \frac{\partial F_2}{\partial q_2}$, $Q_1 = \frac{\partial F_2}{\partial P_1}$ & $Q_2 = \frac{\partial F_2}{\partial P_2}$ $\Rightarrow F_2 = P_1(q_1 - 5q_2) + P_2 q_2$

4. Assuming the transformation is linear

Let $\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = A \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = B \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

$$\Rightarrow \frac{\partial (Q,P)}{\partial (q,p)} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad \left(\frac{\partial (Q,P)}{\partial (q,p)}\right)^T = \begin{pmatrix} A^T & 0 \\ 0 & B^T \end{pmatrix}$$

So
$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} J_n \begin{pmatrix} A^T & 0 \\ 0 & B^T \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & B^T \\ -A^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & AB^T \\ -BA^T & 0 \end{pmatrix}$$

So if $BA^T = I_{n\times n}$ and $AB^T = I_{n \times n}$ $\Rightarrow$ transformation is canonical

But $AB^T = I_{n\times n} \Rightarrow B^T = A^{-1}$ or $B = (A^{-1})^T$

Then $BA^T = (A^{-1})^T A^T = I^T = I$ also

So $B = (A^{-1})^T$ gives a canonical transformation.

5. $H = I_1^2 + I_2 + \epsilon(I_1+I_2)\cos\theta_1 + \epsilon I_2^2$

If $\epsilon = 0$ $\quad H_0(I_1, I_2) = I_1^2 + I_2$ is a function of the actions.

Consider a near identity transformation to remove the order ϵ terms $\quad H_1(I_1, I_2, \theta_1) = (I_1+I_2)\cos\theta_1 + I_2^2$

given by the generating function

$$F_2(\theta_i, J_i) = J_1\theta_1 + J_2\theta_2 + \epsilon\, g(\theta_i, J_i)$$

where $I_i = \dfrac{\partial F_2}{\partial \theta_i} = J_i + \epsilon\dfrac{\partial g}{\partial \theta_i}$ and $\phi_i = \dfrac{\partial F_2}{\partial J_i}$

To transform to A-A. variables the new Hamiltonian $K(\phi_i, J_i)$ must be a function of J_i only.

Since $H_0(I_i) = I_1^2 + I_2 = \left(J_1 + \epsilon\dfrac{\partial G}{\partial \theta_1}\right)^2 + \left(J_2 + \epsilon\dfrac{\partial G}{\partial \theta_2}\right)$

and $H_1(J_i, \theta_1) = (J_1 + J_2)\cos\theta_1 + J_2^2$

$$K = J_1^2 + 2\epsilon J_1\frac{\partial G}{\partial \theta_1} + J_2 + \epsilon\frac{\partial G}{\partial \theta_2} + \epsilon\left((J_1+J_2)\cos\theta_1 + J_2^2\right) + O(\epsilon^2)$$

So we can choose G to remove the oscillating term

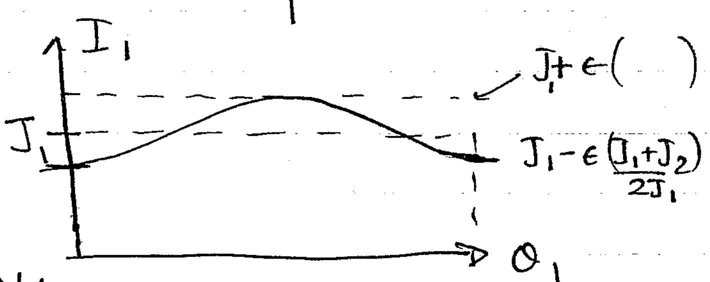
by setting $\quad 2J_1\dfrac{\partial G}{\partial \theta_1} + \dfrac{\partial G}{\partial \theta_2} + (J_1+J_2)\cos\theta_1 = 0$.

$\Rightarrow$ Let $G = -\dfrac{(J_1+J_2)\sin\theta_1}{2J_1}$, then the new Hamiltonian to $O(\epsilon)$ is a function of J_i only

$K(J_i) = J_1^2 + J_2 + \epsilon J_2^2 \Rightarrow J_i$ are constants of the motion.

$\Rightarrow I_1 = J_1 - \dfrac{\epsilon(J_1+J_2)\cos\theta_1}{2J_1}$

is slightly distorted by the perturbing Hamiltonian H_1.



Note also that the frequency $\dfrac{\partial K}{\partial J_2} = 1 + 2\epsilon J_2$ is also changed.

6. The resonance conditions for an oscillating term of the form $\sin(m\theta_1 + n\theta_2)$ or $\cos(m\theta_1 + n\theta_2)$ is

$$m\frac{\partial H_0}{\partial I_1} + n\frac{\partial H_0}{\partial I_2} = 0 \qquad \text{or} \qquad m\omega_1 + n\omega_2 = 0$$

If $H_0 = I_1^2 + 3I_1 I_2 \Rightarrow \omega_1 = 2I_1 + 3I_2$ and $\omega_2 = 3I_1$

Now $\cos 2\theta_1 \sin 2\theta_2 = \frac{1}{2}\left(\sin 2(\theta_1+\theta_2) + \sin 2(\theta_1 - \theta_2)\right)$

Resonances at $2\omega_1 \pm 2\omega_2 = 0$

$$\Rightarrow \quad 2I_1 + 3I_2 = \pm 3I_1 \quad \Rightarrow \quad \underline{3I_2 = I_1 \text{ and } 5I_1 + 3I_2 = 0.}$$

Primary

Also $\cos^2\theta_1 = \frac{1}{2}(\cos 2\theta_1 + 1) \Rightarrow$ Resonances at $\omega_1 = 0$

$$\Rightarrow \underline{2I_1 + 3I_2 = 0}$$

To investigate 2nd order resonances assume that the primary resonances conditions are NOT satisfied and so the resonance terms can be removed by a near identity generating function

$$F_2 = J_1\theta_1 + J_2\theta_2 + \epsilon\left(g_1 \cos 2(\theta_1+\theta_2) + g_2 \cos 2(\theta_1 - \theta_2) + g_3 \sin 2\theta_1\right)$$

$$\Rightarrow I_1 = J_1 - \epsilon\left(g 2\sin 2(\theta_1+\theta_2) + 2g\cos 2(\theta_1-\theta_2) - g_3 2\cos 2\theta_1\right)$$

$$I_2 = J_2 - \epsilon\left(g 2\sin 2(\theta_1+\theta_2) - 2g \sin 2(\theta_1 - \theta_2)\right)$$

This introduces new resonant terms at second order of the form

$$\sin 2(\theta_1+\theta_2)\sin 2(\theta_1-\theta_2) = \frac{1}{4}(\cos 4\theta_1 + \cos 4\theta_2)$$

$$\sin 2(\theta_1+\theta_2)\cos 2\theta_1 = \frac{1}{2}\left(\sin 2(2\theta_1+\theta_2) + \sin 2\theta_2\right)$$

$$\sin(2\theta_1 - 2\theta_2)\cos 2\theta_1 = \frac{1}{2}\sin 2(2\theta_1 - \theta_2) - \sin 2\theta_2$$

$\Rightarrow$ New resonance conditions

$$2\omega_1 \pm \omega_2 = 0 \quad \text{and} \quad -\omega_2 = 0$$

$$\Rightarrow \quad 2(2I_1 + 3I_2) \pm 3I_1 = 0 \quad \text{and} \quad I_1 = 0$$

$$\Rightarrow \quad 7I_1 + 6I_2 = 0 \quad \text{and} \quad I_1 + 6I_2 = 0$$

7 a) $H(\theta_i, I_i) = 2I_1 + I_2 + \epsilon\,(I_2^2 + I_1 I_2^2 \cos(\theta_1 - \theta_2) + I_2 \sin\theta_2)$

The resonance condition for an oscillating term of the form $\cos(n_1\theta_1 + n_2\theta_2)$ is $n_1\omega_1 + n_2\omega_2 = 0$

where $\omega_1 = \frac{\partial H}{\partial I_1} = 2$ and $\omega_2 = \frac{\partial H}{\partial I_2} = 1 \Rightarrow 2n_1 + n_2 = 0$

Since H_0 is linear ω_1 & ω_2 are constants $\Rightarrow$ resonances conditions are always satisfied or never satisfied.

For $\cos(\theta_1 - \theta_2)$ the resonance condition is <u>never satisfied</u>.

$\cos\theta_2$ " " " <u>"</u> .

However there will be new second order resonances with oscillating terms of the form $\cos(\theta_1 - \theta_2)\sin\theta_2 = \sin(\theta_1 - 2\theta_2) + \sin\theta_1$

The resonance conditions are $(2 - 1 \neq 0 \ \text{ or } \ 2 = 0)$

again not satisfied!

b) $H(\theta_i, I_i) = I_1 + 2I_2 + \epsilon\,\frac{I_1}{2}(\cos(2\theta_1 - \theta_2) + \cos\theta_2)$

Primary resonance condition for a term of the form $\cos(n_1\theta_1 + n_2\theta_2)$ is $n_1\omega_1 + n_2\omega_2 = 0$

Here $\omega_1 = 1$ & $\omega_2 = 2 \Rightarrow n_1 + 2n_2 = 0$.

So the term $\cos(2\theta_1 - \theta_2)$ is always resonant

$\Longrightarrow H_0$ is intrinsically degenerate.

So this term cannot be removed by a near identity generating function.

8 a) $H(\theta, I) = H_0(I) + \epsilon I \cos \omega t$

Consider a near identity generating function

$$F_2(\theta, J, t) = \theta J + \epsilon G(\theta, J, t)$$

$$\Rightarrow \quad I = \frac{\partial F_2}{\partial \theta} = J + \epsilon \frac{\partial G}{\partial \theta} \quad \text{and} \quad \phi = \frac{\partial F_2}{\partial J} = \theta + \epsilon \partial G$$

Using Taylor series

$$H_0(I) = H_0(J) + \frac{\partial H_0(J)}{\partial I} \epsilon \frac{\partial G}{\partial \theta} + O(\epsilon^2)$$

$\Rightarrow$ the new Hamiltonian is

$$K(\phi(\theta, J, t), J, t) = H(\theta, I(\theta, J)) + \frac{\partial F_2}{\partial t}$$

$$= H_0(J) + \epsilon \left(\frac{\partial H_0}{\partial I}(J) \frac{\partial G}{\partial \theta} + \frac{\partial G}{\partial t} + J \cos \omega t \right) + O(\epsilon^2)$$

So if we choose $G = -\frac{J}{\omega} \sin \omega t$ then to first order in ϵ

$$K(\phi, J, t) = H_0(J) \quad \Rightarrow \text{the system is a function of the action only.}$$

Hence $J = I - \epsilon \frac{\partial G}{\partial \theta} = I$ is the action.

b) If $\omega(J) \neq -\omega$ the method describe above works with

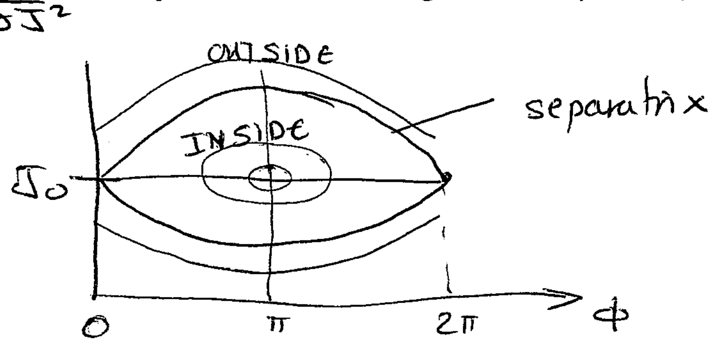
$$G(\theta, J, t) = \frac{-J \sin(\omega t + \theta)}{\omega(J) + \omega}$$

However if $\omega(J_0) = -\omega$ for some J_0 the resonant term cannot be removed. Then transforming to rotating coordinates via $F_2 = J(\theta + \omega t) \Rightarrow \phi = \theta + \omega t$, $J = I$ gives rise to pendulum like behaviour near $J = J_0$.

In fact if $J = J_0 + \sqrt{\epsilon}\, \Delta J$ the new Hamiltonian

$$\bar{H} = H_0(J_0) + \epsilon \left(\frac{\partial^2 H_0}{\partial J^2}(J_0) \Delta J^2 + J_0 \cos \phi \right) + O(\epsilon^2)$$

has a separatrix and different action angle variables must be sought inside the separatrix and outside the separatrix. Also, as with the pendulum, these action angle variables will involve elliptic functions!



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Let $\phi_1 = \theta_1 - \frac{3\theta_2}{2}$ and $\phi_2 = \theta_2$

be new angle variables. Then using an F_2 generating function $F_2(\theta_i, J_i)$; $I_i = \frac{\partial F_2}{\partial \theta_i}$, $\phi_i = \frac{\partial F_2}{\partial J_i}$

$\frac{\partial F_2}{\partial J_1} = \theta_1 - \frac{3\theta_2}{2}$ and $\frac{\partial F_2}{\partial J_2} = \theta_2 \Rightarrow F_2 = J_1(\theta_1 - \frac{3\theta_2}{2}) + J_2\theta_2$

So $I_1 = \frac{\partial F_2}{\partial \theta_1} = J_1$ and $I_2 = \frac{\partial F_2}{\partial \theta_2} = J_2 - \frac{3}{2}J_1$

$\Rightarrow J_1 = I_1$ and $J_2 = I_2 + \frac{3}{2}I_1$

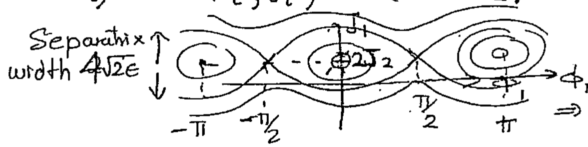
Now $H(\phi_i, J_i) = J_1^2 + 2J_1(J_2 - \frac{3}{2}J_1) + (J_2 - \frac{3}{2}J_1)^2 + \epsilon\cos 2\phi_1$

Since ϕ_2 is not explicitly present $\frac{\partial H}{\partial \phi_2} = 0$

$\Rightarrow \dot{J}_2 = 0$ ie J_2 is a constant of the motion

$\Rightarrow \underline{\underline{I_2 + \frac{3}{2}I_1}}$ "

a) $H(\phi_i, J_i) = (J_2 - J_{1/2})^2 - \epsilon\cos 2\phi_1$



Separatrix given by $H(\pi/2, 2J_2) = \epsilon$

$\Rightarrow \epsilon = (J_2 - J_{1/2})^2 - \epsilon\cos 2\phi_1$

So at $\phi_1 = 0$ $(J_2 - J_{1/2}) = \pm\sqrt{2\epsilon}$

and the separatrix width $= \underline{4\sqrt{2\epsilon}}$

<u>The Poincaré Map / surface of section $\theta_2 = 0$ will be a stroboscopic version of the (ϕ_1, J_1) phase portrait for J_2 held fixed.</u>

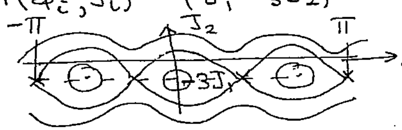
b) If $\phi_1 = \theta_1$ and $\phi_2 = \theta_2 - \frac{2}{3}\theta_1$

$F_2(\theta_i, J_i) = J_1\theta_1 + J_2(\theta_2 - \frac{2}{3}\theta_1)$

and $I_1 = J_1 - \frac{2}{3}J_2$, $I_2 = J_2$ $\Rightarrow J_1 = I_1 + \frac{2}{3}I_2$ constant (as before.)

$H(\phi_i, J_i) = (J_1 + \frac{1}{3}J_2)^2 - \epsilon\cos 3\phi_2$

<u>Poincaré Map SofS $\theta_1 = 0$</u>



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Choose a canonical transformation

$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = A\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \Rightarrow \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = B\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ where $B = (A^{-1})^T$

such that

$q_1^2 + q_2^2 + 2kq_1q_2$ has <u>no cross terms</u> in Q_1, Q_2

And $p_1^2 + p_2^2$ " " " $P_1 P_2$

then, in the new variables (Q_i, P_i), the Hamiltonian will be separable.

Recall if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $B = (A^{-1})^T = \frac{1}{\det A}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

$\Rightarrow B^{-1} = A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Now $p_1^2 + p_2^2$

$= (aP_1 + cP_2)^2 + (bP_1 + dP_2)^2$

$= (a^2+b^2)P_1 + (c^2+d^2)P_2 + 2(ac+bd)P_1P_2$

So we require $ac + bd = 0$ ①

Also $q_1^2 + q_2^2 + 2kq_1q_2$

$= \frac{1}{(\det A)^2}\left((dQ_1 - bQ_2)^2 + (-cQ_1 + aQ_2)^2 + 2k(dQ_1 - bQ_2)(-cQ_1 + aQ_2)\right)$

$= \frac{1}{(ad-bc)^2}\left((d^2 + c^2 - 2cdk)Q_1^2 + (b^2 + a^2 - 2abk)Q_2^2 + (2k(ad+bc) - 2db - 2ac)Q_1Q_2\right)$

So require $2k(ad+bc) - 2(ac+db) = 0$ ②

① & ② $\Rightarrow$ $ac + db = 0$ and $ad + bc = 0$

So now choose something simple that satisfies

eg $a = 1$, $b = +1$, $c = -1$, $d = 1$

$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $\Rightarrow$ $Q_1 = q_1 + q_2$ $Q_2 = -q_1 + q_2$

$B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $P_1 = \frac{1}{2}(p_1 + p_2)$ $P_2 = \frac{1}{2}(-p_1 + p_2)$

Then $q_1^2 + q_2^2 + 2kq_1q_2 = \left(\frac{Q_1 - Q_2}{2}\right)^2 + \left(\frac{Q_1 + Q_2}{2}\right)^2 + 2k\frac{(Q_1^2 - Q_2^2)}{4}$

$$= \frac{Q_1^2}{2}(k+1) + \frac{Q_2^2}{2}(1-k)$$

$$p_1^2 + p_2^2 = \frac{(P_1 - P_2)^2}{4} + \frac{(P_1 + P_2)^2}{4} = \frac{P_1^2}{2} + \frac{P_2^2}{2}$$

So $H = \frac{P_1^2}{4} + \frac{P_2^2}{4} + \frac{Q_1^2}{4}(k+1) + \frac{Q_2^2}{4}(1-k)$

$$= H_1 + H_2$$

where $H_1(Q_1, P_1) = \frac{P_1^2}{4} + \frac{Q_1^2}{4}(k+1)$

and $H_2(Q_2, P_2) = \frac{P_2^2}{4} + \frac{Q_2^2}{4}(1-k)$

Both simple harmonic oscillators.

for action angle variables are polar type coords.

$P_1 = \alpha\sqrt{I_1}\cos\theta_1 \qquad Q_1 = \beta\sqrt{I_1}\sin\theta_1$ for some α & β.

Now $\{Q_1 P_1\}_{\{\theta_1, I_1\}} = 1 \Rightarrow \beta\sqrt{I_1}\cos\theta_1 \frac{\alpha}{2\sqrt{I_1}}\cos\theta_1 + \frac{\beta}{2\sqrt{I_1}}\alpha\sin^2\theta_1 = 1$

$$\Rightarrow \frac{\alpha\beta}{2} = 1$$

Also $H_1 = \left(\frac{\alpha^2}{4}\cos^2\theta_1 + \frac{\beta^2}{4}(k+1)\sin^2\theta_1\right)I_1$ must be a function of I_1 only

so $\beta = \frac{\alpha}{\sqrt{k+1}} \Rightarrow \frac{\alpha^2}{2\sqrt{k+1}} = 1 \Rightarrow \alpha = \sqrt{2}(k+1)^{1/4}$

$$\beta = \frac{\sqrt{2}}{(k+1)^{1/4}}.$$

$P_1 = \sqrt{2I_1}(k+1)^{1/4}\cos\theta_1 \qquad Q_1 = \frac{\sqrt{2I_1}}{(k+1)^{1/4}}\sin\theta_1$

$P_2 = \sqrt{2I_2}(1-k)^{1/4}\cos\theta_2 \qquad Q_2 = \frac{\sqrt{2I_2}}{(1-k)^{1/4}}\sin\theta_2$

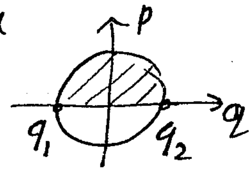
where $I_1 = \frac{2H_1}{\sqrt{k+1}}$ and $I_2 = \frac{2H_2}{\sqrt{1-k}}$

Q11 a) Find action angle variables for the linear oscillator

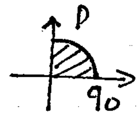
$$H_0 = \tfrac{1}{2}(p^2 + \omega^2 q^2)$$

Assume $H(I)$ then by conservation of area

$$2\pi I = \int dp\,dq = 2\int_{-q_1}^{q_2} p\,dq$$



$$= 4\int_0^{q_0} \sqrt{2H(I) - \omega^2 q^2}\,dq$$



where $H(I) = \frac{\omega^2 q_0^2}{2}$

$$\Rightarrow I = \frac{2\sqrt{2H}}{\pi}\int_0^{q_0} \sqrt{1 - \left(\frac{q}{q_0}\right)^2}\,dq \qquad \text{let } \frac{q}{q_0} = \sin u \Rightarrow u=0 \Rightarrow q=0,\ u = \pi/2 \Rightarrow q = q_0$$

$$\frac{dq}{q_0} = \cos u\,du$$

$$= \frac{2}{\pi}\sqrt{2H}\int_0^{\pi/2} q_0 \cos^2 u\,du$$

$$= \frac{2\times 2H}{\pi\ 2\omega}\int (\cos 2u + 1)\,du = \frac{2H(I)}{\pi\omega}\left(\frac{\sin 2u}{2} + u\right]_0^{\pi/2} = \frac{H}{\omega}$$

$$\Rightarrow H = I\omega$$

Now the generating function $S(q, I)$ satisfies

$$p = \frac{\partial S}{\partial q} \quad \text{and} \quad \theta = \frac{\partial S}{\partial I}$$

$$\Rightarrow S = \int_0^q \sqrt{2H(I) - \omega^2 q^2}\,dq$$

$$\frac{\partial S}{\partial I} = \frac{\partial S}{\partial H}\frac{\partial H}{\partial I} = \omega\int_0^q \frac{dq}{\sqrt{2H(I) - \omega^2 q^2}}$$

Let $\frac{\omega q}{\sqrt{2H}} = \sin u$ then $dq = \frac{\sqrt{2H}}{\omega}\cos u\,du$

$$\Rightarrow \frac{\partial S}{\partial I} = \int du = \arcsin\frac{\omega q}{\sqrt{2H}}$$

$$\Rightarrow q = \frac{\sqrt{2H}}{\omega}\sin\theta \qquad \text{then } p = \sqrt{2H}\cos\theta$$

$$= \sqrt{\frac{2I}{\omega}}\sin\theta \qquad\qquad = \sqrt{2I\omega}\cos\theta$$

Q11

b) In there new variables H

$$\bar{H}(\theta, I) = I\omega + \epsilon\beta\left(\sqrt{\frac{2I}{\omega}}\sin\theta\right)^4$$

$$= I\omega + \frac{\epsilon\beta}{\omega^2} 4I^2 \sin^4\theta$$

Now $\sin^2\theta = \frac{\cos 2\theta - 1}{2} \Rightarrow \sin^4\theta = \frac{\cos^2 2\theta - 2\cos 2\theta + 1}{4}$

$$= \frac{\cos 4\theta - 4\cos 2\theta + 3}{8}$$

$$\bar{H}(\theta, I) = I\omega + \frac{4\epsilon\beta}{\omega^2}\frac{I^2}{8}(\cos 4\theta - 4\cos 2\theta + 3)$$

Now both the oscillating terms are not resonant, provided $\omega \neq 0$ so a near identity transformation should be able to remove them. $(\theta, I) \to (\phi, J)$

So choose an F_2 generating function which for $\epsilon = 0$ is the identity:

$$F_2(\theta, J) = \theta J + \epsilon G(\theta, J)$$

Then $I = \frac{\partial F_2}{\partial \theta} = J + \epsilon \frac{\partial G}{\partial \theta}$

$$\Rightarrow \bar{H}(\theta, I(J, \theta)) = \left(J + \epsilon \frac{\partial G}{\partial \theta}\right)\omega + \frac{\epsilon\beta}{2\omega^2}\left(J + \epsilon\frac{\partial G}{\partial \theta}\right)^2(\cos 4\theta - \dots)$$

$$= \omega J + \epsilon\left(\omega\frac{\partial G}{\partial \theta} + \frac{\beta J^2}{2\omega^2}(\cos 4\theta - 4\cos 2\theta + 3)\right) + O(\epsilon$$

Choose

$$\Rightarrow \omega \frac{\partial G}{\partial \theta} + \frac{\beta J^2}{2\omega^2}(\cos 4\theta - 4\cos 2\theta) = 0$$

Note we cannot remove the nonoscillating term without introducing a term linear in θ which will not remain bounded!

Q11 b) continued

$$\Rightarrow \quad G(\theta, J) = \frac{-\beta J^2}{2\omega^3}\left(\frac{\sin 4\theta}{4} - 2\sin 2\theta\right)$$

Then $I = J + \epsilon \frac{\partial G}{\partial \theta} = J - \frac{\epsilon\beta}{2\omega^3} J^2 (\cos 4\theta - 4\cos 2\theta)$

and $\phi = \frac{\partial F_2}{\partial J} = \theta + \epsilon \frac{\partial G}{\partial J}$

$$= \theta - \frac{\epsilon\beta}{2\omega^2} 2J \left(\frac{\sin 4\theta}{4} - 2\sin 2\theta\right)$$

and $\bar{H}(J) = \omega J + \epsilon \frac{3\beta}{2\omega^2} J^2 + O(\epsilon^2)$

is only a function of J to first order in epsilon.

So J is a constant of the motion $J = \frac{H}{\omega} - \epsilon \frac{3\beta H^2}{2\omega^5}$

$\Rightarrow \dot{\phi} = \frac{\partial \bar{H}}{\partial J} = \omega + \frac{\epsilon 3\beta J}{\omega^2} \approx \omega + \frac{3\beta\epsilon}{\omega^2}\frac{H}{\omega} + O(\epsilon^2)$

$\Rightarrow \phi = \left(\omega + \frac{3\beta\epsilon H}{\omega^3}\right)t + \phi_0$ to order ϵ.

c) Recall $q = \sqrt{\frac{2I}{\omega}} \sin\theta \qquad p = \sqrt{2I\omega}\cos\theta$

where $I = J - \frac{\epsilon\beta}{2\omega^3} J^2 (\cos 4\phi - 4\cos 2\phi) + O(\epsilon^2)$

and $\theta = \phi + \frac{\epsilon\beta J}{\omega^2}\left(\frac{\sin 4\phi}{4} - 2\sin 2\phi\right) + O(\epsilon^2)$

keeping terms to order ϵ

$$I(H,\omega,t) = \frac{H}{\omega} - \epsilon\frac{3\beta H^2}{2\omega^5} + \frac{\epsilon\beta}{2\omega^3}\left(\frac{H}{\omega}\right)^2 \left(\cos 4(\omega t+\phi_0) - 4\cos 2(\omega t+\phi_0)\right)$$

$$\theta(H,\omega,t) = \left(\omega + \frac{3\beta\epsilon H}{\omega^3}\right)t + \phi_0 + \frac{\epsilon\beta}{\omega^2}\left(\frac{H}{\omega}\right)\left(\frac{\sin 4(\omega t+\phi_0)}{4} - 2\sin 2(\omega t+\phi_0)\right)$$