MATH 4104 Tute Sheet 3.

1. $H = I_1 I_2 + \frac{I_2^2}{2} + \epsilon I_1^2 I_2 \cos(4\theta_1 - 2\theta_2)$
$+ \epsilon I_1 I_2 \cos(3\theta_1 - \theta_2)$

Primary resonances occur at

$$4\frac{\partial H}{\partial I_1} - 2\frac{\partial H}{\partial I_2} = 4I_2 - 2(I_1 + I_2) = 0$$
$$\Rightarrow \quad I_1 = I_2$$

and at

$$3\frac{\partial H}{\partial I_1} - \frac{\partial H}{\partial I_2} = 3I_2 - (I_1 + I_2) = 0$$
$$\Rightarrow \quad I_1 = 2I_2$$

To investigate the resonance at $I_1 = I_2$, remove the other resonant term using a near identity generating function. Then transform to new canonical variables.
So if the first near identity transformation transforms (θ_i, I_i) to $(\bar\theta_i, \bar I_i)$ to give

$$\bar H = \bar I_1 \bar I_2 + \frac{\bar I_2^2}{2} + \epsilon \bar I_1^2 \bar I_2 \cos(4\bar\theta_1 - 2\bar\theta_2)$$

Now let $\psi_1 = \bar\theta_1 - \frac{\bar\theta_2}{2}$ and $\psi_2 = \bar\theta_2$

$$\Rightarrow F_2(\bar\theta_i, J_i) \quad \text{satisfies} \quad \psi_1 = \frac{\partial F_2}{\partial J_1} = \bar\theta_1 - \bar\theta_2/2$$
$$\psi_2 = \frac{\partial F_2}{\partial J_2} = \bar\theta_2$$

$$\Rightarrow F_2 = \left(\bar\theta_1 - \frac{\bar\theta_2}{2}\right) J_1 + \bar\theta_2 J_2$$

$$\Rightarrow \bar I_1 = \frac{\partial F_2}{\partial \bar\theta_1} = J_1 \quad \text{and} \quad \bar I_2 = \frac{\partial F_2}{\partial \bar\theta_2} = J_2 - J_1/2$$

In these new variables the approximate Hamiltonian is

$$\bar H(\psi_i, J_i) = J_1(J_2 - J_1/2) + (J_2 - J_1/2)^2/2$$
$$+ \epsilon J_1^2 (J_2 - J_1/2) \cos 4\psi_1 \qquad + O(\epsilon^2)$$

Now J_2 is a constant of the motion

$$H(\psi_i, J_i) = J_1(J_2 - J_{1/2}) + (J_2 - J_{1/2})^2/2 + \epsilon J_1^2 (J_2 - J_{1/2}) \cos 4\psi_1$$
$$= (J_2 - J_{1/2})(3/4 J_1 + J_{2/2}) + \epsilon \quad ''$$

Now resonance occurs at $I_1 = I_2$

$$\Rightarrow J_1 = J_2 - J_{1/2} \Rightarrow J_2 = 3/2 J_1$$

So let $J_1 = 2/3 J_2 + \Delta J_1$ where ΔJ_1 is small

Then
$$(J_2 - J_{1/2})(3/4 J_1 + J_{2/2}) = \left(\tfrac{2}{3}J_2 - \tfrac{\Delta J_1}{2}\right)\left(\tfrac{J_2}{2} + \tfrac{J_2}{2} + 3/4 \Delta J_1\right)$$
$$= \tfrac{2}{3}\left(J_2 - 3/4 \Delta J_1\right)\left(J_2 + 3/4 \Delta J_1\right)$$
$$= 2/3 \left(J_2^2 - 9/16 (\Delta J_1)^2\right)$$

$$\Rightarrow H = 2/3 J_2^2 - \tfrac{3}{8} \Delta J_1^2 + \epsilon (2/3 J_2)^3 \cos 4\psi_1$$

So taking ΔJ_1 to be order $\sqrt{\epsilon}$
the width of the separatrix is $2 . \sqrt{(2/3 J_2)^3 \frac{16\epsilon}{3}}$

$$= 8 \sqrt{\tfrac{I_1^3 \epsilon}{3}}$$ in original actions

in (θ, I_1) space.

Alternatively use the formula.

$$H_0(J_1, J_2) = (J_2 - J_{1/2})\left(\tfrac{3J_1}{4} + \tfrac{J_2}{2}\right) = \tfrac{J_2^2}{2} + \tfrac{J_1 J_2}{2} - \tfrac{3}{8} J_1^2$$

so $\frac{\partial^2 H_0}{\partial J_1^2} = -3/4$

and the coef of the resonant oscillation is $\epsilon I_1^2 I_2$

So at resonace $= \epsilon I_1^3 =$

$$\Rightarrow \text{separatrix width} = 2\sqrt{\epsilon} \sqrt{\left|\frac{4V(I)}{\frac{\partial^2 H}{\partial J_1^2}(I)}\right|}$$

$$= 2 \times \sqrt{\epsilon \times \tfrac{16}{3} I_1^3} = 8 \sqrt{\tfrac{\epsilon I_1^3}{3}}$$

For the second resonace $I_1 = 2I_2$

let $\psi_1 = \bar{\theta}_1 - \bar{\theta}_2/3$, $\psi_2 = \bar{\theta}_2$

$$\Rightarrow F_2 = (\bar{\theta}_1 - \bar{\theta}_2/3)J_1 + \bar{\theta}_2 J_2$$

$$\Rightarrow \bar{I}_1 = J_1 \quad \text{and} \quad \bar{I}_2 = J_2 - J_1/3$$

$$\Rightarrow H_0 = J_1(J_2 - J_1/3) + (J_2 - J_1/3)^2/2$$

$$= J_2^2/2 + 2/3 J_1 J_2 - 5/8 J_1^2$$

$$\Rightarrow \frac{\partial^2 H_0}{\partial J_1^2} = -5/9$$

and separatrix width $= 2\sqrt{\epsilon}\sqrt{\frac{4 \times 9}{5} I_1 \times I_1/2}$

$$= \frac{6\sqrt{\epsilon} I_1}{\sqrt{10}}$$

OR width $= 4\sqrt{\epsilon}\sqrt{\frac{9}{5}\left(\frac{6J_2}{5}\right)^2 \frac{}{2}}$

$$= \frac{72\sqrt{\epsilon} J_2}{5\sqrt{10}}$$

$=$

2. a) – c) are time depenent Hamiltonian systems with one degree of freedom. They are integrable if one can transform to a time independent Hamiltonian.

For instance a) is integrable because we can transform to a rotating frame: –

let $\phi = \theta + \omega t$ and use an F_2 generating function

$$\Rightarrow \quad \phi = \frac{\partial F_2}{\partial J} = \theta + \omega t \quad \Rightarrow F_2 = J(\theta + \omega t)$$

$$\Rightarrow I = J$$

So the new Hamiltonian is $K = H + \frac{\partial F_2}{\partial t}$

$$K(\phi, J) = J + J^2 \epsilon (\cos\phi + \sin^2\phi) - \omega J$$

which is a constant of the motion.

b) Is also integrable. To see this use an F_2 generating function that transforms to a rotating frame and also removes the time dependent term.

let $\phi = \theta + \frac{\omega t}{2} \quad \Rightarrow F_2 = J\left(\theta + \frac{\omega t}{2}\right) + h(t)$

$$\Rightarrow K = J^2 + \epsilon \cos 2\phi + \epsilon \sin 2\omega t + \frac{\partial h}{\partial t}$$

So choose $h = \frac{\epsilon \cos 2\omega t}{2}$ then

$K = J^2 + \epsilon \cos 2\phi$ is integrable.

c) This last one is also integrable because

$$\cos\theta \sin(\theta + \omega t) = \sin(2\theta + \omega t) + \sin \omega t$$

so as above we can transform to a rotating frame and also include a term to remove the time dependence.

2 d) - h) are time independent two degree of freedom Hamiltonian systems. One method for proving integrability is to transform to action angle variables.

Alternatively since H is already a constant of the motion if a second constant can be found the system is integrable

d) is integrable because I_2 is a constant of the motion

$$\dot{I}_2 = \frac{\partial H}{\partial \theta_2} = 0 .$$

e) is also integrable because you can transform to new coordinates

$$\phi_1 = \theta_1 - 3\theta_2 \quad \text{and} \quad \phi_2 = \theta_2$$

via $F_2 = (\theta_1 - 3\theta_2)J_1 + \theta_2 J_2$

in which $H(J_i, \phi_1)$ is not a function of ϕ_2

so again $\dot{J}_2 = \frac{\partial H}{\partial \phi_2} = 0 \Rightarrow J_2$ is a second constant of the motion.

f) is also integrable because if you transform to new coordinates

$$\phi_1 = 5\theta_1 - 2\theta_2 \quad , \phi_2 = \theta_1 + \theta_2$$

via $F_2 = (5\theta_1 - 2\theta_2)J_1 + (\theta_1 + \theta_2)J_2$

then $I_1 = 5J_1 + J_2 \qquad I_2 = -2J_1 + J_2$

$$\Rightarrow H = 3J_1 + J_2 + \cos\phi_1 + \cos\phi_2$$
$$= H_1(\phi_1, J_1) + H_2(\phi_2, J_2) \quad \text{is separable}$$

so both H_1 & H_2 are constants of the motion

g) is not integrable. However KAM tori may not exist for $\epsilon = 0$ because the resonances may not be isolated. The Hessian

$$\det \begin{pmatrix} \frac{\partial^2 H_0}{\partial I_1^2} & \frac{\partial^2 H_0}{\partial I_1 \partial I_2} \\ \frac{\partial^2 H_0}{\partial I_2 \partial I_1} & \frac{\partial^2 H_0}{\partial I_2^2} \end{pmatrix} = \det \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 0$$

h) is not integrable and KAM tori will exist:

The Hessian $= \det \begin{pmatrix} 0 & 2I_2 \\ 2I_2 & I_1 \end{pmatrix} = -4I_2^2 \neq 0$

Primary resonaces occur at $\omega_1 = 2\omega_2 \Rightarrow I_2^2 = 2 \times 2 I_1 I_2$
and at $\omega_2 = 0 \Rightarrow I_1 I_2 = 0$

Secondary " " " $\omega_1 = 3\omega_2$ and $\omega_1 = \omega_2$
etc

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3. a) $a_0 = [\frac{21}{13}] = 1 \Rightarrow \omega_1 = \frac{13}{8}$ and $a_1 = [\frac{13}{8}] = 1$

$\omega_2 = \frac{8}{5} \Rightarrow a_2 = 1$, $\omega_3 = 5/3 \Rightarrow a_3 = 1$,

$\omega_4 = 3/2 \Rightarrow a_4 = 1$, $\omega_5 = 2 \Rightarrow a_5 = 2$ and $a_6 = \infty$

$\frac{12}{13} = [1, 1, 1, 1, 1, 2, \infty]$

b) Now $2 < \sqrt{5} < 3$

$a_0 = 1 \Rightarrow \omega_1 = \frac{1}{\frac{47-\sqrt{5}}{38} - 1} = \frac{38}{9-\sqrt{5}} = \frac{9+\sqrt{5}}{2}$

$a_1 = 5 \Rightarrow \omega_2 = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}$

$a_2 = 1 \Rightarrow \omega_3 = \frac{2}{\sqrt{5}-1}$ and so the process will repeat itself ad infinitum

$\Rightarrow a_n = 1$ for $n \geq 2$.

$\frac{47-\sqrt{5}}{38}$ $[1, 5, 1, 1, \dots]$

c) $x_\pm = 1 \pm \sqrt{2}$

For $x_+ = 1 + \sqrt{2}$ $\quad a_0 = 2$, $\omega_1 = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{1}$

and so the process will repeat

$x_+ = 1 + \sqrt{2} = [2, 2, 2, \dots]$

For $x_- = 1 - \sqrt{2} < 0$

So consider $\sqrt{2} - 1$ for which $a_0 = 0$

$\omega_1 = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{1} = x_+$

$\Rightarrow x_- = -[0, 2, 2 \dots]$

d) $\gamma = \frac{1+\sqrt{5}}{2}$ $\quad a_0 = [\frac{1+\sqrt{5}}{2}] = 1$

$\omega_1 = \frac{1}{\frac{1+\sqrt{5}}{2} - 1} = \frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{4} = \frac{\sqrt{5}+1}{2} = \gamma$

So $\omega_n = \frac{\sqrt{5}+1}{2}$ $\forall n \geq 1$ and $a_n = 1$ $\forall n \geq 0$

4. Since the map is area preserving $\det T = 1$
We are given that $\operatorname{trace} T = \pm 2$
So eigen values given by $\lambda^2 + \operatorname{trace} T\,\lambda + \det T$
$$\lambda^2 \pm 2\lambda + 1 = 0$$
$$\Rightarrow \lambda_+ = -1 \text{ twice} \quad \lambda_- = 1 \text{ twice}$$
So in normal form $u_{n+1} = \pm u_n$, $v_{n+1} = \pm v_n$
In (+) case all points are fixed.
(−) " " " are period-2.
except the origin which is fixed.

5. See notes for this one.

6. $\frac{1}{\gamma^2} = \frac{1}{\left(\frac{1+\sqrt{5}}{2}\right)^2} = \frac{3-\sqrt{5}}{2} \Rightarrow a_0 = 0$
$$\omega_1 = \frac{2}{3-\sqrt{5}} = \frac{3+\sqrt{5}}{2} \Rightarrow a_1 = 2$$
$$\omega_2 = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2} \Rightarrow a_2 = 1$$
and $a_n = 1$ for $n \geq 2$

$$\frac{1}{\gamma^2} = [0, 2, 1, 1, 1 \ldots]$$
Rational Approximates are $[0, \infty] = 0$,
$$[0, 2, \infty] = \frac{1}{2+\frac{1}{\infty}} = \tfrac{1}{2},$$
$$[0, 2, 1, \infty] = \frac{1}{2+\frac{1}{1+\frac{1}{\infty}}} = \frac{1}{3},$$
$$[0, 2, 1, 1, \infty] = \frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{\infty}}}} = \tfrac{2}{5},$$
$$[0, 2, 1, 1, 1, \infty] = \tfrac{3}{8},$$
If $K \approx 0$ then $I_{n+1} \approx I_n$ and winding number $= \frac{I_n}{2\pi}$
So $I_n = 0, \frac{1}{4\pi}, \frac{1}{6\pi}, \frac{2}{10\pi}$ and $\frac{3}{16\pi}$ when $K = 0$.

7a) If I_1 and I_2 are involutions of a map so that $I_1^2 = I_2^2$ = identity and if $\underset{\sim}{x}$ and $(I_2 I_1)^n \underset{\sim}{x}$ are fixed points of I_1 so that

$$I_1 \underset{\sim}{x} = \underset{\sim}{x} \quad \text{and } I_1 (I_2 I_1)^n \underset{\sim}{x} = (I_2 I_1)^n \underset{\sim}{x}$$

then

$$\begin{aligned}
(I_2 I_1)^{2n} \underset{\sim}{x} &= (I_2 I_1)^{n-1} (I_2 I_1)(I_2 I_1)^n \underset{\sim}{x} \\
&= (I_2 I_1)^{n-1} I_2 (I_2 I_1)^n \underset{\sim}{x} && (I_1 (I_2 I_1)^n \underset{\sim}{x} = (I_2 I_1)^n \underset{\sim}{x}) \\
&= (I_2 I_1)^{n-1} I_2^2 I_1 (I_2 I_1)^{n-1} \underset{\sim}{x} \\
&= (I_2 I_1)^{n-1} I_1 (I_2 I_1)^{n-1} \underset{\sim}{x} && (I_2^2 = \text{identity}) \\
&= (I_2 I_1)^{n-2} I_2 I_1^2 (I_2 I_1)^{n-1} \underset{\sim}{x} \\
&= (I_2 I_1)^{n-2} I_2 (I_2 I_1)^{n-1} \underset{\sim}{x} && I_1^2 = \text{identity} \\
&\;\vdots \\
&= I_2 (I_2 I_1) \underset{\sim}{x} \\
&= I_1 \underset{\sim}{x} = \underset{\sim}{x}
\end{aligned}$$

So that $\underset{\sim}{x}$ is a period $2n$ point of the map.

b). The involutions of the map are given by

$$I_1 \begin{pmatrix} \theta \\ J \end{pmatrix} = \begin{pmatrix} -\theta \\ J + f(\theta) \end{pmatrix} \Rightarrow I_1^2 \begin{pmatrix} \theta \\ J \end{pmatrix} = \begin{pmatrix} -(-\theta) \\ J + f(\theta) + f(-\theta) \end{pmatrix} = \begin{pmatrix} \theta \\ J \end{pmatrix}$$

since $f(-\theta) = -f(\theta)$

and

$$I_2 \begin{pmatrix} \theta \\ J \end{pmatrix} = \begin{pmatrix} -\theta + g(J) \\ J \end{pmatrix} \Rightarrow I_2^2 \begin{pmatrix} \theta \\ J \end{pmatrix} = \begin{pmatrix} -(-\theta + g(J)) + g(J) \\ J \end{pmatrix} = \begin{pmatrix} \theta \\ J \end{pmatrix}$$

Also

$$I_2 I_1 \begin{pmatrix} \theta \\ J \end{pmatrix} = I_2 \begin{pmatrix} -\theta \\ J + f(\theta) \end{pmatrix} = \begin{pmatrix} \theta + g(J + f(\theta)) \\ J + f(\theta) \end{pmatrix}$$

as required.

7c) Now suppose that $f(\theta) = \sin\theta$ and $g(J) = J^2$.

$$(I_2 I_1)\begin{pmatrix}\theta \\ J\end{pmatrix} = \begin{pmatrix}\theta + (J + \sin\theta)^2 \\ J + \sin\theta\end{pmatrix}$$ where θ is mod 2π

Now the fixed lines of I_1 are

$\theta = -\theta + 2n\pi$ and $f(\theta) = 0 \Rightarrow \sin\theta = 0$

$\Rightarrow \theta = n\pi$ //

and of I_2 are

$2\theta = g(J) = J^2$

$\Rightarrow \quad \theta = J^2/2$ and $\theta = \pi + J^2/2$.

So consider a point on the fixed line of I_1 that maps to the other fixed line of I_1.

That is a point $\begin{pmatrix}0 \\ J\end{pmatrix}$ s.t $I_2 I_1 \begin{pmatrix}0 \\ J\end{pmatrix}$ lies on $\begin{pmatrix}\pi \\ J'\end{pmatrix}$

Now $I_2 I_1 \begin{pmatrix}0 \\ J\end{pmatrix} = \begin{pmatrix}J^2 \\ J\end{pmatrix}$ So this means that $\begin{pmatrix}0 \\ J\end{pmatrix}$ where $J^2 = \pi$ or $J = \pm\sqrt{\pi}$ should be a period-2 pt.

Also the point $\begin{pmatrix}\sqrt{\pi} \\ \pi\end{pmatrix} = I_2 I_1 \begin{pmatrix}0 \\ \pm\sqrt{\pi}\end{pmatrix}$ should be a period-2 pt.

Check! $(I_2 I_1)^2 \begin{pmatrix}0 \\ \pm\sqrt{\pi}\end{pmatrix} = I_2 I_1 \begin{pmatrix}\pi \\ \pm\sqrt{\pi}\end{pmatrix} = \begin{pmatrix}\pi + (\pm\sqrt{\pi} + \sin\pi)^2 \\ \pm\sqrt{\pi} + \sin\pi\end{pmatrix} = \begin{pmatrix}0 \\ \pm\sqrt{\pi}\end{pmatrix}$

since θ is mod 2π.

Q8 a) Suppose that the continued fraction representation of x is $[\overline{a_0 a_1 \ldots a_n}]$

then $x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$ $\Rightarrow$ $\omega_1 = \frac{1}{x - a_0} = a_1 + \cfrac{1}{a_2 + \cdots}$

and $\omega_2 = \cfrac{1}{\frac{1}{x - a_0} - a_1} = \frac{x - a_0}{1 - a_1(x - a_0)}$ is a ratio of polynomials of degree 1 with integer coefficients.

In fact if ω_j is a ratio of polynomials of degree 1 with integer coefficients $\Rightarrow$ $\omega_{j+1} = \frac{1}{\omega_j - a_j}$ also has this property:

So if $\omega_j = \frac{b_1 x + b_2}{b_3 x + b_4}$ $\Rightarrow$ $\omega_{j+1} = \frac{b_3 x + b_4}{b_1 x + b_2 - a_j(b_3 x + b_4)}$

Hence $\omega_{n+1} = \frac{c_1 x + c_2}{c_3 x + c_4}$ for some integers c_i, $i = 1 \ldots 4$.

$$\omega_{n+1} = x \Rightarrow x = \frac{c_1 x + c_2}{c_3 x + c_4}$$

$$\Rightarrow c_3 x^2 + (c_4 - c_1)x - c_2 = 0$$

ie x satisfies a quadratic with integer coefficients.

b) If the continued fraction representation of x is $[a_0 \ldots a_m \overline{a_{m+1} \ldots a_n}]$

then ω_{n+1} has continued fraction representation $[\overline{a_{m+1} \ldots a_n}]$

So ω_{n+1} satisfies a quadratic with integer coefficients.

But also $\omega_{n+1} = \frac{c_1 x + c_2}{c_3 x + c_4}$ for some integers c_i, $i = 1 \ldots 4$.

$$\Rightarrow d_1 \left(\frac{c_1 x + c_2}{c_3 x + c_4}\right)^2 + d_2 \left(\frac{c_1 x + c_2}{c_3 x + c_4}\right) + d_3 = 0$$

$$(d_1 c_1^2 + d_2 c_3 c_1 + d_3 c_3^2)x^2 + (2d_1 c_1 c_2 + d_2(c_1 c_4 + c_3 c_4) + 2d_3 c_3 c_4)x$$
$$+ d_1 c_2^2 + d_2 c_2 c_4 + d_3 c_4^2 = 0$$