— REPLACE WITH COVER PAGE —

Instructions

The exam consists of 5 questions, 1– 5. Each question has three items, a – c.

Within each question:

Item (a) carries a weight of 10 marks.

Item (b) carries a weight of 6 marks.

Item (c) carries a weight of 4 marks.

Answer ALL questions in the spaces provided.

If more space is required, use the back of the PREVIOUS page.

Show all your working and include sketches where appropriate.

Question 1:

For every integer $n \geq 1$, denote $\mathcal{N}_n = \{1, \ldots, n\}$. Further, denote by $\mathcal{P}_n = 2^{\mathcal{N}_n}$ the power-set of $\mathcal{N}_n$ comprised of all subsets of $\mathcal{N}_n$. For example,

$$\mathcal{P}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

(1a) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{|\mathcal{P}_n|},$$

or if the series does not converge, explain why.

(1b) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{|\mathcal{N}_n|},$$

or if the series does not converge, explain why.

(1c) For $k \in \{0, \ldots, n\}$, consider the sets,

$$\mathcal{A}_n = \{q \in \mathcal{P}_n \mid |q| = k\},$$

and,

$$\mathcal{B}_n = \{q \in \mathcal{P}_n \mid |q| = n - k\}.$$

Determine $|\mathcal{A}_n| - |\mathcal{B}_n|$ and prove your result.

Question 2:

Let n and k be positive integers with $k \leq n$. Consider the identity (sometimes known as Pascal's formula),

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

(2a) Prove this identity using only words (without writing any mathematical symbols except for n and k). If you are not able to do that, prove it using mathematical symbols.

(2b) Use induction and the above identity to prove that for $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(2c) Prove (in any way possible) that for $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Question 3:

Given a function, $f : [0, \infty) \to \mathbb{R}$ and a variable $s \in \mathbb{R}$, it is often of interest to evaluate the function, $\widehat{f}(s)$ defined as

$$\widehat{f}(s) = \int_0^\infty e^{sx} f(x)\, dx,$$

when the indefinite integral converges.

(3a) Let $f(x) = 4e^{-4x}$. Determine $\widehat{f}(8)$.

(3b) Continuing with $f(x) = 4e^{-4x}$, for what values of $s \in \mathbb{R}$ does the indefinite integral of $\widehat{f}(s)$ converge?

(3c) For an arbitrary $f(x)$, denote,

$$m_n = \int_0^\infty x^n f(x)\, dx.$$

Assuming you may interchange limits of summations and integrals, use the fact that,

$$e^{sx} = \sum_{k=0}^{\infty} \frac{(sx)^k}{k!},$$

to show that for $n = 0, 1, 2, \ldots$

$$\frac{d\widehat{f}(s)^n}{ds^n}\Big|_{s=0} = m_n.$$

Here the expression means taking the n'th derivative of $\widehat{f}(s)$ and then substituting $s = 0$.

Question 4:

Consider the function,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}}.$$

It is known that,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

(4a) Does $f(x)$ have a global maximum? If so what is it? Prove/show your result using the first and second derivative.

(4b) Evaluate,

$$\int_{-\infty}^{\infty} x f(x)\, dx.$$

(4c) Determine an interval $[a, b] \subset \mathbb{R}$ such that,

$$[a, b] = \{x \in \mathbb{R} \mid f''(x) \leq 0\}.$$

If no such interval exists, prove it.

Question 5:

Consider the matrix

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}.$$

Denote by $A^n = A \times A \times \ldots \times A$ (matrix multiplication of n matrices).

(5a) Calculate A^3.

(5b) It is believed that,

$$(*) \qquad A^n = \frac{1}{3}\Big(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \frac{1}{4^n} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \Big).$$

Verify that (*) is true for $n = 1, 2, 3$.

(5c) Prove that (*) holds for all $n \geq 1$ by induction.

END OF EXAMINATION